Math 221, Section 1.6

Applications
(1) Equilibrium in economics
(2) Network flow
Leontief’s input-output model

- Suppose a nation’s economy is divided into many sectors (such as various manufacturing, communication, entertainment, and service industries.)
- Suppose we also know, for each sector,
  - its total output for one year and
  - how this output is divided among the other sectors and itself of the economy.
- Leontief proved the following result.
  There exists an equilibrium price that can be assigned to the output of each sector in a way that balances the input of each sector exactly.
Consider an economy with three sectors: Fuels and Power, Manufacturing, and Services.

- Fuels and Power sells 80% of its output to Manufacturing, 10% to Services, and retains the rest.
- Manufacturing sells 10% of its output to Fuels and Power, 80% to Services, and retains the rest.
- Services sells 20% to Fuels and Power, 40% to Manufacturing, and retains the rest.

Exchange table

<table>
<thead>
<tr>
<th></th>
<th>F&amp;P</th>
<th>Man</th>
<th>Ser</th>
<th>Purchased by</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.2</td>
<td>0.8</td>
<td>F&amp;P</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.4</td>
<td></td>
<td>Man</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.8</td>
<td></td>
<td>Ser</td>
<td></td>
</tr>
</tbody>
</table>
Example (Sec 1.6, Ex 3)

Let

- $x_{out}/x_{in}$ be the output/input of Fuels and Power,
- $y_{out}/y_{in}$ be the output/input of Manufacturing,
- $z_{out}/z_{in}$ be the output/input of Services

Output/Input equations

\[
0.1x_{out} + 0.1y_{out} + 0.2z_{out} = x_{in} \\
0.8x_{out} + 0.1y_{out} + 0.4z_{out} = y_{in} \\
0.1x_{out} + 0.8y_{out} + 0.4z_{out} = z_{in}
\]
Example (Sec 1.6, Ex 3)

(a) Develop a system of equations that leads to prices (of the output) at which each sector’s income matches its expenses (of the inputs).

At equilibrium prices, $x_{out} = x_{in}$, $y_{out} = y_{in}$, $z_{out} = z_{in}$.

The equations become

\[
\begin{align*}
0.1x + 0.1y + 0.2z &= x \\
0.8x + 0.1y + 0.4z &= y \\
0.1x + 0.8y + 0.4z &= z
\end{align*}
\]

Right hand sides are variables, need to rewrite as

\[
\begin{align*}
-0.9x + 0.1y + 0.2z &= 0 \\
0.8x - 0.9y + 0.4z &= 0 \\
0.1x + 0.8y - 0.6z &= 0
\end{align*}
\]
(b) Then write the augmented matrix that can be row reduced to find the equilibrium prices.

Apply Gaussian elimination

\[
\begin{bmatrix}
-0.9 & 0.1 & 0.2 \\
0.8 & -0.9 & 0.4 \\
0.1 & 0.8 & -0.6
\end{bmatrix} \rightarrow (Exercise) \rightarrow \begin{bmatrix}
1 & 0 & -\frac{22}{73} \\
0 & 1 & -\frac{52}{73} \\
0 & 0 & 0
\end{bmatrix}
\]

Equation now reads: \( x - \frac{22}{73}z = 0 \), \( y - \frac{52}{73}z = 0 \), and \( z \) is a parameter

If take \( z = 73t \), where \( t \) is a parameter, then

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
\frac{22t}{52} \\
\frac{52t}{73} \\
\frac{73t}{73}
\end{bmatrix} = t \begin{bmatrix}
\frac{22}{52} \\
\frac{52}{73}
\end{bmatrix}.
\]

That means: if you can maintain the of outputs at the ratio \( x : y : z = 22 : 52 : 73 \), then the economy is at equilibrium.
Example (Sec 1.6, Ex 3)

(c) Find a set of equilibrium prices when the price for the Services output (variable z) is 100 units.

Answer: When $z = 100$, then

\[ x = 100 \times \frac{22}{73} \approx 30.137 \]
\[ y = 100 \times \frac{52}{73} \approx 71.233 \]
Network flow

- Systems of linear equations arise naturally when scientists, engineers, or economists study the flow of some quantity through a network (often involve hundreds or even thousands of variables and equations!!).
- A network consists of
  - a set of points called junctions, or nodes,
  - with arrowed lines or arcs called branches connecting some or all of the junctions with directions indicated, and
  - the flow amount (or rate) is either a given constant or a variable.
- A basic assumption: the flow into a junction equals the flow out of the junction. (So the total flow is ”conserved” in the network.) For example, Fig. 1 shows $x_1 + x_2 = 30$. 

![Figure 1](image-url)
EXAMPLE 2 The network in Fig. 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

FIGURE 2 Baltimore streets.
Example (Sec 1.6, Ex 12)

- (a) Find the general flow pattern of the network shown in the figure.

- Tabularize

<table>
<thead>
<tr>
<th>Junction</th>
<th>Flow in</th>
<th>Flow out</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$x_1 + x_4$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>B</td>
<td>$x_2$</td>
<td>$x_3 + 100$</td>
</tr>
<tr>
<td>C</td>
<td>$x_3 + 80$</td>
<td>$x_4$</td>
</tr>
</tbody>
</table>
Example (Sec 1.6, Ex 12)

- Rewrite

\[
\begin{align*}
x_1 + x_4 &= x_2 \\
x_2 &= x_3 + 100 \\
x_3 + 80 &= x_4
\end{align*}
\]

as

\[
\begin{align*}
x_1 - x_2 + x_4 &= 0 \\
x_2 - x_3 &= 100 \\
x_3 - x_4 &= -80
\end{align*}
\]

- Apply Gaussian elimination

\[
\begin{bmatrix}
1 & -1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 & 100 \\
0 & 0 & 1 & -1 & -80
\end{bmatrix}
\rightarrow (Exercise) \rightarrow 
\begin{bmatrix}
1 & 0 & 0 & 0 & 20 \\
0 & 1 & 0 & -1 & 20 \\
0 & 0 & 1 & -1 & -80
\end{bmatrix}
\]

- The equation reads \( x_1 = 20, \ x_2 - x_4 = 20, \ x_3 - x_4 = -80, \) with \( x_4 \) as a parameter, or we can write

\[
\begin{align*}
x_1 &= 20 \\
x_2 &= x_4 + 20 \\
x_3 &= x_4 - 80
\end{align*}
\]
Sometimes the flows of a network are one-way, that means none of the variables can be negative. This limits the possible values of the variables.

(b) Assuming that the flows are one-way (so that they must be all non-negative), what is the smallest possible value for $x_4$?

Answer: If $x_3$ is non-negative, it means that $x_3 \geq 0$. Since $x_3 = x_4 - 80$, we have

$$x_4 - 80 \geq 0 \quad \Rightarrow \quad x_4 \geq 80.$$  

This means $x_4$ cannot be too small; otherwise the system won’t start.
Suggested exercises for Quiz 2 (again, questions similar to bold ones will appear in the quiz)

- Sec 1.4: 3, 4, 7, 11, 14, 15, 16, 17, 19, 21, 22, 23b, c, e, f, 24b, c, d, f, 26
- Sec 1.5: 1, 3, 5, 7, 11, 13, 15, 17, 23a, d, e, 24a, b, d, e
- Sec 1.6: 1, 3, 5, 12, 13, 14

You may check your solution at