Math 221 - Sec 205 - Winter 2017 - Midterm 2

Solution

Name: Student #: "

Instructions

(1) Put all your personal belongings at the front of the exam room.

(2) Bring your student ID card with you and put it on the table.

(3) You are prohibited from bringing any electronic devices into exam table or in the pockets, whether the devices are turned off or not. Violators will not only be asked to leave the exam room, but will also be suspected of attempted cheating.

(4) You are prohibited from bringing any course materials, scrap papers, or calculators.

(5) When time is up, stop writing and hand in the paper immediately. Your information should be filled in at the beginning of the test.

(6) Violators of any of the above instructions will be immediately disqualified and reported to the department.

(7) If you observe any misconduct or attempted cheating, report to the instructor or the TA immediately.
(1) (2 marks) Given a $3 \times 3$ matrix $A$ with columns $a_1, a_2, a_3$ such that
\[
a_1 + a_2 = a_3,
\]
Determine whether each of the following statements is true or false. (No work is required.)

(i) The null space of $A$ is non-zero.

**True.** It is because $a_1 + a_2 - a_3 = 0$ is the same as $A \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = 0$, which implies that \[
\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}
\]
is in the null space of $A$.

(ii) The columns of $A$ spans $\mathbb{R}^3$.

**False.** Because $\text{span}\{a_1, a_2, a_3\} = \text{span}\{a_1, a_2\}$, and 2 vectors can never span $\mathbb{R}^3$ which is 3 dimensional.

(iii) The RREF of $A$ is the identity matrix.

**False.** If the RREF of $A$ is the identity matrix, then $Ax = 0$ has only trivial solution, but we have seen in (i) that it has a non-trivial solution.

(iv) $\det A = 0$.

**True.** It is because
\[
\det[a_1, a_2, a_3] = \det[a_1, a_2, a_1 + a_2] = \det[a_1, a_2, a_2] + \det[a_1, a_2, a_2].
\]
The last equality is due to the linear property. Now observe that in each of the last two matrices, there are two identical columns, so that its determinant must be 0.

**Remark.** All these questions can be also answered using the Invertible Matrix Theorem (IMT). Indeed all statements here are the falsehood of those in the IMT, which means that the following statements are equivalent.

- $e'$. The column vectors are **not** linearly independent,
- $q'$. The null space is **not** the zero subspace,
- $h'$. The column vectors does **not** span $\mathbb{R}^n$,
- $b'$. The RREF of $A$ is **not** $I$,
- $t'$. $\det A = 0$. 

(2) (1 mark) Concerning determinants, no work is required.

(a) If $A$ is a $4 \times 4$ matrix and $\det A = 1$, what is $\det(2A)$?

**Solution:** $\det(2A) = 2^4 \det A = 16$.

(b) If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 1$, what is $\det \begin{bmatrix} p & q & r \\ x & y & z \\ a & b & c \end{bmatrix}$?

**Solution:** Since

$$
\begin{aligned}
\begin{bmatrix}
 p & q & r \\
 x & y & z \\
 a & b & c 
\end{bmatrix} &\xrightarrow{R_2 \leftrightarrow R_3} \\
&\xrightarrow{R_1 \leftrightarrow R_2} \\
\begin{bmatrix}
 p & q & r \\
 a & b & c \\
 x & y & z 
\end{bmatrix}
\end{aligned}
$$

we applied two row interchangings, the determinant is $(-1)^2 = 1$. 

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(3) (4 marks) A $2 \times 4$ matrix $A$ with columns $a_1, \ldots, a_4$ is reduced to the following RREF.

\[
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 2 & 1
\end{bmatrix}
\]

If $a_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $a_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, what is $a_4$?

**Solution:** The RREF above tells us that

\[
a_3 = a_1 + 2a_2 \quad \text{and} \quad a_4 = 2a_1 + a_2.
\]

We can use the first relation to solve for $a_1$:

\[
a_1 = a_3 - 2a_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\]

and so we use the second relation and obtain

\[
a_4 = 2a_1 + a_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.
\]

**Alternative solution:** If you don’t want to compute $a_1$, you can directly row-reduce the matrix obtained by deleting the first column of RREF:

\[
\begin{bmatrix}
0 & 1 & 2 \\
1 & 2 & 1
\end{bmatrix} R_2 \leftrightarrow R_1 \quad \Rightarrow \quad \begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 2
\end{bmatrix} R_1 - 2R_2 \quad \Rightarrow \quad \begin{bmatrix}
1 & 0 & -3 \\
0 & 1 & 2
\end{bmatrix}.
\]

That means

\[
a_4 = -3a_2 + 2a_3 = -3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.
\]

Same answer as in the solution above.
(4) (6 marks)

(a) Suppose \( x \) is a vector in \( \mathbb{R}^2 \), and \( \mathcal{B} = \{ \mathbf{b}_1, \mathbf{b}_2 \} \) is a basis for \( \mathbb{R}^2 \). The coordinate vector \([x]_{\mathcal{B}}\) consists of entries \( \begin{bmatrix} x \\ y \end{bmatrix} \) which appears in the linear combination

\[
x = x \mathbf{b}_1 + y \mathbf{b}_2.
\]

Rewrite this vector equation as a matrix multiplication relating \( x \), \([x]_{\mathcal{B}}\), and the change-of-coordinate matrix \( P_{\mathcal{B}} \). In particular, write down how the matrix \( P_{\mathcal{B}} \) is formed.

**Solution:** The relation is given by

\[
x = P_{\mathcal{B}} [x]_{\mathcal{B}}
\]

and

\[
P_{\mathcal{B}} = [\mathbf{b}_1, \mathbf{b}_2].
\]

(b) In the following diagram,

\[
\begin{array}{c}
\xrightarrow{T} \quad T(x) \\
\downarrow \quad \downarrow \\
[x]_{\mathcal{B}} \quad \xrightarrow{[T]_{\mathcal{B}}^C} \quad [T(x)]_C
\end{array}
\]

write down the matrices representing the vertical arrows. (Be careful of the directions.)

**Solution:**

\[
\begin{array}{c}
\xrightarrow{T} \quad T(x) \\
\downarrow \quad P_{\mathcal{B}} \quad \downarrow \quad P_{\mathcal{C}} \\
[x]_{\mathcal{B}} \quad \xrightarrow{[T]_{\mathcal{B}}^C} \quad [T(x)]_C
\end{array}
\]
(c) If \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is a linear transformation such that
\[
T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.
\]

Find the standard matrix of \( T \).

**Solution:** you may use the following fact:

The \( j \)-th column of \( [T]_{\mathcal{E}}^B \) is given by \( [T(b_j)]_C \).

In our question, since \( C \) is just the standard basis \( \mathcal{E} \), the statement is simplified into:

The \( j \)-th column of \( [T]_{\mathcal{E}}^B \) is given by \( T(b_j) \).

Therefore, we immediately have
\[
[T]_{\mathcal{E}}^B = \begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix}.
\]

We also have the change-of-coordinate matrices
\[
P_B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{E}} = I.
\]

Then we apply the transition formula (direct from part (b))
\[
T = [T]_{\mathcal{E}}^B P_B^{-1} = \begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 7 \\ -1 & 3 \end{bmatrix}.
\]

**Alternatively,** if you just want the calculation without knowing any principle, then you can just put the two equations in the question together as
\[
T \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 1 \\ 3 \end{bmatrix}.
\]

Then multiplying \( \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} \) on both sides no the right and get
\[
T = \begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1},
\]
which eventually is the same as above.
(5) (4 marks) Compute the determinant of the following $4 \times 4$ matrix

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
2 & 3 & 1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

using cofactor expansion.

**Solution:** Observe that the last column has all zero entries except the last one, we expand along the last column and get

\[
\det A = (-1)^{4+4} \cdot 1 \cdot \det \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}.
\]

From here you can use Gaussian elimination or cofactor expansion. For example, you may expand along the first row and get

\[
\det \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = (-1)^{1+1} \cdot 1 \cdot \det \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} + (-1)^{1+2} \cdot 1 \cdot \det \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}
\]

\[
= 3 \cdot 2 - 1 \cdot 1 - (2 \cdot 2 - 1 \cdot 0) = 1.
\]

**Answer:** $\det A = 1$. 


(6) (4 marks) A plane in $\mathbb{R}^3$ passes through the following three points,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}.$$ Does this plane form a subspace in $\mathbb{R}^3$? Explain.

(Hint: What if it does, then can you tell any relation between these vectors?)

**Solution:** These 3 vectors form a plane, that is a 2-dimensional subspace in $\mathbb{R}^3$, if and only if they are linearly dependent. There are two ways to check this.

- Using Gaussian elimination, we can check

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 3 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 3 & 1 & 4 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ The last column is non-pivotal, that means the columns are linearly dependent.

- Using determinant, either using Gaussian elimination (which is just given by above) or cofactor expansion, one can check that the determinant is 0. This also implies that the columns are linearly dependent.
(7) (4 marks) In the following determinant of a $3 \times 3$ matrix, the first and third columns are fixed and the second column is a variable column.

\[
\begin{vmatrix}
1 & x & 1 \\
1 & y & 2 \\
1 & z & 3 \\
\end{vmatrix}
\]

This defines a linear transformation on the second column. What is the matrix representing this linear transformation?

(Hint: What is the size of this matrix? What is its $j$-th column?)

**Solution:** Denote this linear transformation by

\[ T : \mathbb{R}^3 \rightarrow \mathbb{R}^1, T(\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}) = \det \begin{vmatrix}
1 & x & 1 \\
1 & y & 2 \\
1 & z & 3 \\
\end{vmatrix}
\]

Here the codomain is $\mathbb{R}^1$ which is the codomain of the determinant function. Therefore, the matrix is a $1 \times 3$ matrix.

To find out its $j$-th column, we use the fact that:

The $j$-th column of $T$ is $T(e_j)$.

Therefore, we have to compute $T(e_1), T(e_2), \text{ and } T(e_3)$.

\[
T(e_1) = \det \begin{vmatrix}
1 & 1 & 1 \\
1 & 0 & 2 \\
1 & 0 & 3 \\
\end{vmatrix} = (-1)^{1+2} \det \begin{vmatrix}
1 & 2 \\
1 & 3 \\
\end{vmatrix} = -1
\]

\[
T(e_2) = \det \begin{vmatrix}
1 & 0 & 1 \\
1 & 1 & 2 \\
1 & 0 & 3 \\
\end{vmatrix} = (-1)^{2+2} \det \begin{vmatrix}
1 & 1 \\
1 & 3 \\
\end{vmatrix} = 2
\]

\[
T(e_3) = \det \begin{vmatrix}
1 & 0 & 1 \\
1 & 0 & 2 \\
1 & 1 & 3 \\
\end{vmatrix} = (-1)^{3+2} \det \begin{vmatrix}
1 & 1 \\
1 & 2 \\
\end{vmatrix} = -1
\]

We use the cofactor expansion along the second column in all calculations above (because all entries except one are zeros). The matrix of $T$ is therefore

\[
\begin{bmatrix}
-1 & 2 & -1 \\
\end{bmatrix}
\]