March 11, 2020

Math 318 Assignment 8: Due Friday, March 20 at start of class

Test 2 will be held in class on Monday March 23, and will be based on the material covered in Assignments 5–8. No assignment is due March 25. Assignment 9, which is the last one, will be available March 25.

I. Problems to be handed in:

1. A random number of people enter an elevator on the ground floor, with the distribution of the number being Poisson(λ). There are N floors above the ground floor and each person is equally likely to get off at any one of the N floors. Assume that the elevator does not make stops to pick up passengers until all who entered on the ground floor have gotten off. Show that the expected number of stops that the elevator will make before all passengers get off is $N(1-e^{-\lambda/N})$.
   Hint: the indicator random variable I_i, which is 1 if the elevator stops at floor i and otherwise is 0, may be useful. Condition on the initial number of passengers.

2. Consider simple symmetric random walk in two dimensions.

   (a) Show that the probability p_{2n} that a walker returns to its starting place at the origin after $2n$ steps is given by $p_{2n} = \frac{1}{4^{2n}}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2$.

   (b) Show that $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$. Hint: Think of choosing n balls from a box that has $2n$ balls, n white and n black.

   (c) Conclude that $p_{2n} \sim \frac{1}{\pi n}$ as $n \to \infty$.

   (d) Conclude that the two-dimensional walk is recurrent.

   Note: the combination of parts (a) and (b) shows that the 2-dimensional return probability is the square of its 1-dimensional counterpart. An explanation of this is presented in Section 2.3.6 of Doyle and Snell `https://arxiv.org/abs/math/0001057`.

3. For a random walk in d dimensions taking i.i.d. steps, we have seen that the expected number of visits to the origin is given by

$$m = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{1}{1-\phi_1(\vec{k})}\, d^d\vec{k}, \tag{1}$$

   where $\phi_1(\vec{k})$ is the characteristic function of a single step. For $d = 3$, we have discussed the simple random walk on the cubic lattice $\mathbb{Z}^3$ where the six allowed steps are of the form $(\pm1, 0, 0)$, $(0, \pm1, 0)$, $(0, 0, \pm1)$. A random walk on the *face-centred cubic lattice* takes steps of the form $(\pm1, \pm1, 0)$, $(\pm1, 0, \pm1)$, $(0, \pm1, \pm1)$ with equal probabilities $\frac{1}{12}$.

   (a) Show that

$$\phi_1(\vec{k}) = \frac{1}{3}(\cos k_1 \cos k_2 + \cos k_2 \cos k_3 + \cos k_3 \cos k_1)$$

   for the walk on the face-centred cubic lattice. The trigonometric identity $\cos(x+y)+\cos(x-y) = 2\cos x \cos y$ is useful here.

   (b) Deduce that the integrand in (1) has singularities not only for $\vec{k} = \vec{0}$ but at eight additional points on the boundary of the Brillouin zone $[-\pi, \pi]^3$.

   (c) Using a Taylor expansion to second order (neglecting all terms beyond the quadratic), argue that the singularity at $\vec{k} = \vec{0}$ is integrable. (The other singularities are also integrable, leading to the conclusion that $m < \infty$ and hence the walk is transient, but you need not show this.)

4. (a) Suppose that a random walk is recurrent (probability is 1 that it returns to the origin) and that there is a positive probability for the walk to eventually visit some other site x, starting from the origin. Explain why the random walk will eventually visit x infinitely often.

(b) Consider a random walk on $\mathbb{Z}^1$ taking steps 0, +2, −2 with respective probabilities $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$. Compute $\phi_1(k)$, and argue that the integral

$$m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1-\phi_1(k)}dk$$

is infinite, so the walk is recurrent.

(c) Consider two independent walkers performing simple random walk (each taking steps ± 1 with equal probabilities $\frac{1}{2}$) on $\mathbb{Z}^1$, with one walk beginning at 0 and the other at +2. Will the two walkers certainly collide at some point? (Hint: consider the difference between their positions.)

5. This problem concerns return times of simple random walk on $\mathbb{Z}^d$ in dimensions $d = 1, 2$.

Let $S_0, S_1, \ldots$ denote d-dimensional symmetric simple random walk started at the origin. Let $\tau^{(d)} = \min\{n \geq 1 : S_n = 0\}$ be the first return time to the origin. Since the random walk is recurrent for $d = 1, 2$, $\tau^{(d)}$ is a finite random variable. It is known[1] that, as $n \to \infty$,

$$\mathbb{P}\left(\tau^{(d)} > n\right) \sim \begin{cases} \left(\frac{2}{\pi n}\right)^{1/2} & \text{if } d = 1 \\ \pi(\log n)^{-1} & \text{if } d = 2. \end{cases} \tag{2}$$

We will observe this result through simulations in Python. Submit your code and plots and answers to the questions.

Parts (b) and (c) involve performing linear regression in Python. This is explained in the tutorial file available at:
`http://www.math.ubc.ca/~geoff/courses/W2019T2/LR_tutorial.pdf`.

(a) Write a function $\mathsf{tau}(t, d)$ that receives as inputs $t \geq 0$ and $d \in \{1, 2\}$ and runs a simulation of $S_0, \ldots, S_T$, where $T = \min(\tau^{(d)}, t)$, and returns:

$$\begin{cases} \tau^{(d)} & \text{if the random walk returns to the origin before or at time } t \\ \mathsf{float('inf')} & \text{otherwise.} \end{cases}$$

Notes (i) "float('inf')" is a Python command that behaves like "infinity". It is bigger than x for any real number x.
(ii) For the sake of efficiency, it is important that the above function only generates the random walk until time $T = \min(\tau^{(d)}, t)$, rather than always generating it until time t. If necessary, you can use the command break to interrupt a loop.

(b) Call the function $\mathsf{tau}(5000, 1)$ 4000 times; let $r(1), \ldots, r(4000)$ be the results obtained. Define

$$a(n) = \frac{\#\{i : r(i) > n\}}{4000}, \qquad \text{for } n \in \{1, 2, \ldots, 5000\}.$$

On a graph, plot $\log(a(n))$ on the y-axis vs $\log n$ for $n \in \{1, 2, \ldots, 5000\}$ on the x-axis. On the same graph, plot the corresponding line of best fit. The title of your plot should contain the slope obtained in the linear regression; the tutorial file shows how this is done. According to (2), what value should the slope obtained be close to?

(c) Proceed similarly for the 2-dimensional walk: call the function $\mathsf{tau}(5000, 2)$ 4000 times; let $R(1), \ldots, R(4000)$ be the results obtained and define

$$A(n) = \frac{\#\{i : R(i) > n\}}{4000}, \qquad \text{for } n \in \{1, 2, \ldots, 5000\}.$$

On a graph, plot $1/A(n)$ on the y axis vs $\log n$ for $n \in \{1, 2, \ldots, 5000\}$ on the x-axis, and again a line of best fit with its slope in the title of the graph. According to (2), what value should the slope be close to?

[1] This result can be found in Proposition 4.2.4 (p.80) in *Random Walk: A Modern Introduction*, by G.F. Lawler and V. Limic, `http://www.math.uchicago.edu/~lawler/srwbook.pdf`.

II. Recommended problems: These provide additional practice but are not to be handed in.

A. A random walk on the *body-centred cubic lattice* takes steps of the form $(\pm 1, \pm 1, \pm 1)$ (eight possible steps) with equal probabilities $\frac{1}{8}$. Show that $\phi_1(\vec{k}) = \cos k_1 \cos k_2 \cos k_3$ for the walk on the body-centred cubic lattice.

B. Chapter 3: #3 $[2, \frac{5}{3}, \frac{12}{5}]$, 5 $[\binom{3}{i}\binom{6}{3-i}/\binom{9}{3}, \frac{5}{3}]$, 9, 19*.

For pictures of the face-centred cubic lattice and the body-centred cubic lattice, see `http://en.wikipedia.org/wiki/Cubic_crystal_system`, or use google images to see many more.