Total marks = 30.

1. Since, for $a \geq 0$,
\[ P(X > a | X < Y) = \frac{P(a < X < Y)}{P(X < Y)} = \frac{\lambda + \mu}{\lambda} P(a < X < Y) = \frac{\lambda + \mu}{\lambda} \int_a^\infty dx \int_x^\infty dy \lambda e^{-\lambda x} \mu e^{-\mu y} = e^{-(\lambda + \mu)a}, \]
we find that the conditional density is (for $a \geq 0$)
\[ f(a) = \frac{d}{da} P(X \leq a | X < Y) = \frac{d}{da} (1 - e^{-(\lambda + \mu)a}) = (\lambda + \mu)e^{-(\lambda + \mu)a}, \]
so the conditional distribution is Exp($\lambda + \mu$).

2. (a) $N_5$ is Poisson($5\lambda$) so its mean is $5\lambda$.
(b) $S_3$ is Gamma($3, \lambda$) so its mean is $3/\lambda$.
(c) $P(N_5 < 3) = e^{-5\lambda}(1 + 5\lambda + 25\lambda^2/2)$.
(d) If you like doing integrals you can integrate the Gamma density but it is easier to use the fact that $\{S_3 > 5\}$ is the same event as $\{N_5 < 3\}$ so the answer is the same as in part (c), namely: $e^{-5\lambda}(1 + 5\lambda + 25\lambda^2/2)$.
(e) By the no memory property the conditional probability is equal to $P(S_2 > 3)$ and as in (d) and (a) this equals $P(N_5 < 2) = e^{-3\lambda}(1 + 3\lambda)$.

3. (a) The number of orders by time $t$ is $M_t + V_t$, which has a Poisson($\mu(\mu + v)t$) distribution.
(b) For $k = 0, 1, \ldots, n$,
\[ P(M_t = k | M_t + V_t = n) = \frac{P(M_t = k, V_t = n - k)}{P(M_t + V_t = n)} = \frac{P(M_t = k)P(V_t = n - k)}{P(M_t + V_t = n)} = \frac{\mu^k e^{-\mu}(\mu + v)^{n-k} e^{-\mu - v}}{(n-k)!} = \binom{n}{k} \left(\frac{\mu}{\mu + v}\right)^k \left(\frac{v}{\mu + v}\right)^{n-k}. \]
The conditional distribution of the number of meat orders is therefore Bin($n, \mu/\mu+v$).

4. (a) $\phi_S(t) = \phi_{X_1}(t) \cdots \phi_{X_n}(t) = \exp[i \sum_{k=1}^n \mu_k t - \sum_{k=1}^n \sigma_k^2 t^2/2]$, so $S_n \sim N(\mu_1 + \cdots + \mu_n, \sigma_1^2 + \cdots + \sigma_n^2)$.
(b) Let $S = X_1 + X_2 + X_3 - X_4 \sim N(5 + 5 + 5 - 5, 4 + 4 + 4 + 4) = N(10, 16)$ (since $-X_4 \sim N(-5, 4)$ and by (a)), so
\[ P(X_1 + X_2 + X_3 < X_4) = P(S < 0) = P\left(\frac{S-10}{4} < \frac{0-10}{4}\right) = P(Z < -2.5) = P(Z > 2.5) = 1 - \Phi(2.5) = 0.0062 \]
(c) $\phi_{Y_n}(t) = \phi_{S_n}(t/n) = e^{i\mu t} e^{-\sigma^2 t^2/2n}$, so $X \sim N(\mu, \sigma^2/n)$.
Similarly, $\phi_{Z_n}(t) = \phi_{S_n}(t/\sqrt{n}) = e^{i\mu \sqrt{n} t} e^{-\sigma^2 t^2/2}$, so $X \sim N(\mu \sqrt{n}, \sigma^2)$. (In particular, if $\mu = 0$ then $Z_n \sim N(0, \sigma^2)$ has the same distribution as $X_t$ for all $n$).
(d) $\lim_{n \to \infty} \phi_{Y_n}(t) = \lim_{n \to \infty} e^{i\mu t} e^{-\sigma^2 t^2/2n} = e^{i\mu t}$.
(e) With $\mu = 0$ and $\sigma = 1$, by (c) we have $\sqrt{n}Y_n \sim N(0, 1)$, so
\[ P(|Y_n| \leq 0.1) = P(|Z| \leq (0.1)(\sqrt{n})) = \Phi(\sqrt{n}/10) - \Phi(-\sqrt{n}/10) = 2\Phi(\sqrt{n}/10) - 1. \]
The table then gives the values: .0796 ($n = 1$), .1742 ($n = 5$), .5222 ($n = 50$), .9750 ($n = 500$).
5. (a) The left and right derivatives of $\phi$ at $t = 0$ are +1 and −1, respectively, so $\phi$ is not differentiable at $t = 0$.

(b) $\phi_{n^{-1}S_n}(t) = (\phi(t/n))^n = (e^{-|t|/n})^n = e^{-|t|} = \phi(t)$, so $n^{-1}S_n$ has the same distribution as $X_1$, namely a Cauchy distribution.

(c) The weak law of large numbers does not apply to the Cauchy distribution, since its expectation is undefined.

6. (a) The number of earthquakes in 100 decades is a Poisson random variable with parameter $\lambda t = 100$.

The following Jupyter notebook performs the simulations and makes the plots for (b) and (c).

```python
[11]: import numpy as np
import numpy.random as npr

[12]: def quakes(L):
    T = npr.exponential(1);
    y = 0;
    while(T < L):
        y = y + 1;
        T = T + npr.exponential(1);
    return y

[19]: X = np.zeros(10000)
for i in range(10000):
    X[i] = quakes(100)

[35]: import matplotlib.pyplot as plt

# Produce histogram
plt.hist(X,bins=100,range=(50,150));
plt.title("Number of earthquakes in 10000 samples of 100 decades");
plt.show()

# "X==100" is a vector with 0's and 1's; 1's mark the indices j such that X(j) ==\n...100 % "cumsum(X==100)" gives the cumulative sum of this vector,
# as required in the definition of $M$
M = np.cumsum(X==100);

# Finally, create a vector $N$ such that $N(i) = M(i)/i$ and plot it
N = np.zeros(10000)
for i in range(10000):
    N[i] = M[i] / (i+1)
plt.plot(N);
plt.title("Plot of $M(i)/i$")
```
(d) By the Law of Large Numbers, the limiting value should be $P(Y = 100)$, where $Y \sim \text{Poisson}(100)$. 
Thus, using Sterling’s formula,

\[
P(Y = 100) = \frac{100^{100}}{100!} e^{-100} \approx \frac{100^{100}}{100^{100} e^{-100} \sqrt{200\pi} e^{-100}} = 0.040.
\]