Math 318 Assignment 5: Due Wednesday, February 26 at start of class

I. Problems to be handed in:

1. Let $X, Y$ be independent exponential random variables with respective rates $\lambda, \mu$. Find the conditional distribution of $X$ given that $X < Y$.
   
   (You may use the fact that a slight generalisation of Assignment 4 #4(b) shows that if $X, Y$ are independent exponential random variables of respective rates $\lambda, \mu$, then $P(X < Y) = \frac{\lambda}{\lambda + \mu}$.)

2. Let $\{N_t : t \geq 0\}$ be a Poisson process of rate $\lambda$, and let $S_n$ denote the time of the $n$th event.
   
   Find:
   
   (a) $E(N_5)$;
   (b) $E(S_3)$;
   (c) $P(N_5 < 3)$;
   (d) $P(S_3 > 5)$;
   (e) $P(S_3 > 5 \mid N_2 = 1)$.

3. A pizzeria sells meat and vegetarian pizzas. The number of meat orders by time $t$ is a Poisson process $(M_t)_{t \geq 0}$ with rate $\mu$. The number of vegetarian orders is a Poisson process $(V_t)_{t \geq 0}$ with rate $\nu$. These two Poisson processes are assumed to be independent. It is a theorem that the sum of two independent Poisson processes with rates $\mu$ and $\nu$ is itself a Poisson process whose rate is $\mu + \nu$.
   
   (a) What is the distribution of the total number of orders by time $t$?
   (b) Given that $n$ orders are received by time $t$, find and identify the conditional distribution of the number of meat orders that have arrived by time $t$.

4. (a) Suppose that $X_1, X_2, \ldots$ are independent Gaussian random variables, with $X_i \sim N(\mu_i, \sigma_i^2)$. Let $S_n = X_1 + \cdots + X_n$. Compute the characteristic function of $S_n$ and thereby identify its distribution.
   
   (b) Four fish are caught in a day. Their weights (in pounds) are independent $N(5, 4)$ random variables (this is an approximation — in reality the weights cannot be negative). Find the probability that the last fish weighs more than the other 3 together. Hint: Consider $X_1 + X_2 + X_3 - X_4$; if $X_4 \sim N(5, 4)$ then what is the distribution of $-X_4$?
   
   (c) Now assume in (a) that $\mu_i = \mu$ and $\sigma_i = \sigma$ for all $i$. Let $Y_n = n^{-1} S_n$ denote the average of the first $n$ $X_i$’s. Identify the distribution of $Y_n$, by calculating its characteristic function. Do the same for $Z_n = n^{-1/2} \sum_{i=1}^n X_i$.
   
   (d) Show explicitly that the limit, as $n \to \infty$, of the characteristic function of $Y_n$ approaches the characteristic function of a constant random variable (as in the proof of the weak law of large numbers).
   
   (e) Suppose $\mu = 0$ and $\sigma = 1$, so that each $X_i$ is a standard normal random variable. Compare the probabilities that $|Y_n| \leq 0.1$, for $n = 1, n = 5$, $n = 50$ and $n = 500$.

5. The standard Cauchy random variable has probability density function $f(x) = \frac{1}{\pi(1+x^2)}$ and characteristic function $\phi(t) = e^{-|t|}$. Suppose that $X_1, X_2, \ldots$ are independent standard Cauchy random variables and let $S_n = \sum_{i=1}^n X_i$.
   
   (a) We have seen that $EX_1$ is undefined. Check this via the characteristic function.
(b) Use characteristic functions to show that $n^{-1}S_n$ is also a standard Cauchy random variable. (This helps explain what you observed in Assignment 4 #6(b).)

(c) Why does (b) not contradict the weak law of large numbers?

6. Suppose that the number of decades between the occurrence of two serious earthquakes in a region follows an exponential distribution with parameter 1. In parts (b,c), print and submit your code and plots.

(a) Let $Y$ denote the number of earthquakes in a period of 100 decades. What is the distribution of $Y$?

(b) Using Python, generate independent random variables $X_1, \ldots, X_{10000}$, each of which gives the total number of earthquakes that occur in a simulation of a period of 100 decades. Present the results in a histogram which shows the number of 100 decade periods (among the 10000) which produced any given total number of earthquakes.

(c) Using the same simulation as in (b), for $1 \leq i \leq 10000$, let $M(i)$ denote the number of $j \in \{1, \ldots, i\}$ such that $X_j = 100$. Plot a graph of $M(i)/i$ vs $i$, for $1 \leq i \leq 10000$.

(d) In (c), what is the limiting value and why? (You may find it useful to recall Stirling’s formula $n! \approx n^n e^{-n} \sqrt{2\pi n}$.)

II. Recommended problems: These provide additional practice but are not to be handed in.

A. Chapter 5, #3, 57*.

B. Chapter 2, #54 (1, 1, 2p(1, 1) − 1).

C. Let $X$ be distributed geometrically with parameter $p$. Compute the characteristic function of $X$ and use it to show that the variance of $X$ is $\frac{1-p}{p^2}$.

D. Let $X, Y$ be independent exponential random variables with parameters $\lambda, \mu$, respectively. Show that the conditional distribution of $X$, given that $X < Y$, is Exp($\lambda + \mu$).

E. This problem shows that the Weak Law of Large Numbers does not imply the Strong Law of Large Numbers. Let $X, Y$ be independent and each with p.m.f. $p(0) = p(1) = \frac{1}{2}$. Let $X_n = Y$ for all $n = 0, 1, \ldots$. Show that $X_n$ converges in distribution to $X$, but that $P(\lim_{n \to \infty} X_n = X) = \frac{1}{2}$. 

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