February 13, 2020

Math 318 Assignment 5: Due Wednesday, February 26 at start of class

I. Problems to be handed in:

1. Let X, Y be independent exponential random variables with respective rates λ, μ. Find the conditional distribution of X given that $X < Y$.
   (You may use the fact that a slight generalisation of Assignment 4 #4(b) shows that if X, Y are independent exponential random variables of respective rates λ, μ, then $P(X < Y) = \frac{\lambda}{\lambda+\mu}$.)

2. Let $\{N_t : t \geq 0\}$ be a Poisson process of rate λ, and let S_n denote the time of the nth event. Find:
   (a) $E(N_5)$;
   (b) $E(S_3)$;
   (c) $P(N_5 < 3)$;
   (d) $P(S_3 > 5)$;
   (e) $P(S_3 > 5 \mid N_2 = 1)$.

3. A pizzeria sells meat and vegetarian pizzas. The number of meat orders by time t is a Poisson process $(M_t)_{t\geq 0}$ with rate μ. The number of vegetarian orders is a Poisson process $(V_t)_{t\geq 0}$ with rate ν. These two Poisson processes are assumed to be independent. It is a theorem that the sum of two independent Poisson processes with rates μ and ν is itself a Poisson process whose rate is $\mu + \nu$.
   (a) What is the distribution of the total number of orders by time t?
   (b) Given that n orders are received by time t, find and identify the conditional distribution of the number of meat orders that have arrived by time t.

4. (a) Suppose that $X_1, X_2, \ldots$ are independent Gaussian random variables, with $X_i \sim N(\mu_i, \sigma_i^2)$. Let $S_n = X_1 + \cdots + X_n$. Compute the characteristic function of S_n and thereby identify its distribution.
   (b) Four fish are caught in a day. Their weights (in pounds) are independent $N(5, 4)$ random variables (this is an approximation — in reality the weights cannot be negative). Find the probability that the last fish weighs more than the other 3 together.
   Hint: Consider $X_1 + X_2 + X_3 - X_4$; if $X_4 \sim N(5, 4)$ then what is the distribution of $-X_4$?
   (c) Now assume in (a) that $\mu_i = \mu$ and $\sigma_i = \sigma$ for all i. Let $Y_n = n^{-1}S_n$ denote the average of the first n X_i's. Identify the distribution of Y_n, by calculating its characteristic function. Do the same for $Z_n = n^{-1/2}\sum_{i=1}^n X_i$.
   (d) Show explicitly that the limit, as $n \to \infty$, of the characteristic function of Y_n approaches the characteristic function of a constant random variable (as in the proof of the weak law of large numbers).
   (e) Suppose $\mu = 0$ and $\sigma = 1$, so that each X_i is a standard normal random variable. Compare the probabilities that $|Y_n| \leq 0.1$, for $n = 1$, $n = 5$, $n = 50$ and $n = 500$.

5. The standard Cauchy random variable has probability density function $f(x) = \frac{1}{\pi(1+x^2)}$ and characteristic function $\phi(t) = e^{-|t|}$. Suppose that $X_1, X_2, \ldots$ are independent standard Cauchy random variables and let $S_n = \sum_{i=1}^n X_i$.
   (a) We have seen that EX_1 is undefined. Check this via the characteristic function.

(b) Use characteristic functions to show that $n^{-1}S_n$ is also a standard Cauchy random variable. (This helps explain what you observed in Assignment 4 #6(b).)

(c) Why does (b) not contradict the weak law of large numbers?

6. Suppose that the number of decades between the occurrence of two serious earthquakes in a region follows an exponential distribution with parameter 1. In parts (b,c), print and submit your code and plots.

   (a) Let Y denote the number of earthquakes in a period of 100 decades. What is the distribution of Y?

   (b) Using Python, generate independent random variables $X_1, \ldots, X_{10000}$, each of which gives the total number of earthquakes that occur in a simulation of a period of 100 decades. Present the results in a histogram which shows the number of 100 decade periods (among the 10000) which produced any given total number of earthquakes.

   (c) Using the same simulation as in (b), for $1 \le i \le 10000$, let $M(i)$ denote the number of $j \in \{1, \ldots, i\}$ such that $X_j = 100$. Plot a graph of $M(i)/i$ vs i, for $1 \le i \le 10000$.

   (d) In (c), what is the limiting value and why?
   (You may find it useful to recall Stirling's formula $n! \approx n^n e^{-n}\sqrt{2\pi n}$.)

II. Recommended problems: These provide additional practice but are not to be handed in.

A. Chapter 5, #3, 57*.

B. Chapter 2, #54 (1, 1, $2p(1,1) - 1$).

C. Let X be distributed geometrically with parameter p. Compute the characteristic function of X and use it to show that the variance of X is $\frac{1-p}{p^2}$.

D. Let X, Y be independent exponential random variables with parameters λ, μ, respectively. Show that the conditional distribution of X, given that $X < Y$, is $\text{Exp}(\lambda + \mu)$.

E. This problem shows that the Weak Law of Large Numbers does not imply the Strong Law of Large Numbers. Let X, Y be independent and each with p.m.f. $p(0) = p(1) = \frac{1}{2}$. Let $X_n = Y$ for all $n = 0, 1, \ldots$. Show that X_n converges in distribution to X, but that $P(\lim_{n\to\infty} X_n = X) = \frac{1}{2}$.