January 30, 2020

Math 318 Assignment 4: Due Wednesday, February 5 at start of class

Reminder: Test 1 will be held in class on Monday February 10, and will be based on the material covered in Assignments 1–4. No assignment will be given on February 5; Assignment 5 will be available on February 12.

I. Problems to be handed in:

1. A particle of mass 1 g has a random velocity X that is uniformly distributed between 2 cm/s and 3 cm/s.

   (a) Find the cumulative distribution function of the particle's kinetic energy $T = \frac{1}{2}X^2$.

   (b) Find the probability density function of T.

   (c) Determine the mean and standard deviation of T in two ways:
   (i) using the p.d.f. of T from part (b),
   (ii) using the Law of the Unconscious Statistician and the uniform density of X.

2. Let $X_1, X_2, \ldots, X_n$ be independent random variables, each with uniform distribution on $(0, 1)$. Let M be the minimum of these random variables.

   (a) Show that the cumulative distribution function of M is $F_M(x) = 1 - (1-x)^n$, $0 \le x \le 1$.

   (b) Find the probability density function of M.

   (c) Determine the mean and variance of M.

3. Let X be a uniform random variable on $[0, 1]$ and let Y be uniform on $[-1, 0]$. Compute $\mathrm{Cov}(X, X^2)$ and $\mathrm{Cov}(Y, Y^2)$.
   (The first is positive and the second is negative, consistent with the fact that X^2 increases when X increases and X^2 decreases when X decreases, whereas the opposite is true for Y and Y^2.)

4. The time T (in hours past noon) until the arrival of the first taxi has Exponential(5) distribution, and the time B until the first bus is independent with Exponential(3) distribution.

   (a) Write down the joint probability density function of T and B.

   (b) Find the probability that the first taxi arrives before the first bus.

   (c) If you arrive at noon and take the first bus or taxi (whichever arrives first), what is the distribution of your waiting time (give the name and parameter(s))? [Hint: let $X = \min(T, B)$, and find $P(X > y)$.]

5. This question considers uniform random points on the disc $0 \le x^2 + y^2 \le 1$ with radius 1 centred at the origin. Parts (d) and (e) involve programming; the commands from `scipy.stats.uniform` helpful — for example `scipy.stats.uniform.rvs` generates continuous uniform random numbers on a specified interval.

   (a) A point is uniformly chosen in the unit disk $0 \le x^2 + y^2 \le 1$. Find the probability that its distance from the origin is less than r, for $0 \le r \le 1$.

   (b) Compute its expected distance from the origin.

   (c) Let the co-ordinates of the point be (X, Y). Determine the marginal p.d.f. of X. Are X and Y independent?

   (d) One way to generate uniform random points on this disc is to first generate uniform random points on the square with corners $(\pm 1, \pm 1)$, by independently generating their x and y coordinates as uniform random variables on $(-1, 1)$, and then ignore the points outside the unit circle. To visualize this, generate 5000 uniform random points on the square with corners $(\pm 1, \pm 1)$ and create a scatterplot of their locations. Draw a circle with radius one centred at the origin. The points inside the circle are then uniformly random points in the disk.

(e) Another way to represent points in this unit circle is via polar coordinates. We might try naively to generate uniform random points in the circle by first generating a random radius R uniformly between 0 and 1, and then by generating a random angle T uniformly between 0 and 2π. Generate 5000 such random pairs (R, T) and create a scatterplot of the resulting points in the standard xy-plane; that is, so that each pair gives the point $(x, y) = (R\cos T, R\sin T)$. Compare this scatterplot to the one you created in (a). Does this look uniformly random?

(f) By definition, the density of uniformly random points in the circle with respect to polar coordinates is the function $f(r, \theta)$ for which, if A is a subset of the circle,

$$\iint_A f(r, \theta)\, dr\, d\theta = \frac{\text{area of } A}{\pi}.$$

Using your knowledge of multivariate calculus, what must $f(r, \theta)$ be?

6. In this problem, print and submit code and plots, together with written answers to the questions.

(a) Use Python to simulate a standard normal random variable 10000 times and make a plot of the running average. That is, if X_i is the i^{th} simulated value, make a plot of

$$\frac{X_1 + \cdots + X_n}{n}$$

against n for n from 1 to 10000. Does this plot look like it converges to 0?

(b) Consider the model described in class where a spinner is located one unit of distance away from an infinite wall. The angle Y that the pointer makes with the perpendicular to the wall is uniformly distributed over the interval $(-\pi/2, \pi/2)$. The distance X of the point that the spinner points at to the perpendicular is thus $X = \tan Y$.

It was argued in class that X has no expectation (despite the fact that it might look as if it should have expected value 0). To illustrate this, use Python to simulate X 10000 times and make a plot of the running average of the simulated values. Does this plot look like it converges to 0? Does it look like it converges at all?

II. Recommended problems: These provide additional practice but are not to be handed in.

A. Chapter 2: $41[2(n-1)p(1-p)]$, $43[n/(m+1)]$, 50, $51[r/p]$, 57.

B. Chapter 5: 1 $[e^{-1}, e^{-1}]$, 2 $[6/\mu]$, 3, 4 [assume that all service times are independent; $0, \frac{1}{27}, \frac{1}{4}$].

C. *Optional* problem for those interested in quantum mechanics and the uncertainty principle:

Consider a quantum mechanical system in state ψ, where ψ is vector in the Hilbert space consisting of complex-valued square-integrable functions, with inner product $\langle \psi_1, \psi_2 \rangle = \int \psi_1(x)^* \psi_2(x) dx$. Assume that ψ is normalized, so that $\langle \psi, \psi \rangle = \int |\psi(x)|^2 dx = 1$. Observables such as position $\mathcal{X}$ or momentum $\mathcal{P}$ are represented by self-adjoint linear operators on the Hilbert space. The expected value of an observable $\mathcal{A}$ is given by $\langle \psi, \mathcal{A}\psi \rangle = \int \psi(x)^* (\mathcal{A}\psi)(x) dx$. The standard deviation $\sigma(\mathcal{A})$ of a measurement of the observable $\mathcal{A}$ is given by $\sigma(\mathcal{A})^2 = \langle \psi, (\mathcal{A} - \langle \psi, \mathcal{A}\psi \rangle)^2 \psi \rangle$. It is a general mathematical theorem that for any self-adjoint linear operators $\mathcal{A}$ and $\mathcal{B}$, with commutator $[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$,

$$\langle \psi, \mathcal{A}^2 \psi \rangle \langle \psi, \mathcal{B}^2 \psi \rangle \geq \frac{1}{4} |\langle \psi, [\mathcal{A}, \mathcal{B}]\psi \rangle|^2.$$

(You can prove this by adapting the proof of the Cauchy–Schwarz inequality, or see Lemma IV.6.1 of E. Prugovečki, Quantum Mechanics in Hilbert Space, Academic Press, 1971.)

The commutator of the position and momentum operators is $[\mathcal{X}, \mathcal{P}] = \hbar i$. Using the above, show that $\sigma(\mathcal{X})\sigma(\mathcal{P}) \geq \hbar/2$. This is a statement of the uncertainty principle.