Math 318 Solutions to Assignment #3 January 27, 2020

Total marks = **[30]**.

1. Let X denote the player's profit and Y the number of times the player's number appears on the dice. Note that $X = -1$ if $Y = 0$, and that $X = k$ if $Y = k$ for $k = 1, 2, 3$. Also, $Y \sim \text{Bin}(3, \frac{1}{6})$ so $P(Y = k) = \binom{3}{k}(\frac{1}{6})^k(\frac{5}{6})^{3-k}$ for $k = 0, 1, 2, 3$. Then **[6]**

$$EX = (-1)P(X = -1) + \sum_{k=1}^{3} kP(X = k) = -P(Y = 0) + \sum_{k=1}^{3} kP(Y = k)$$
$$= -P(Y = 0) + \sum_{k=0}^{3} kP(Y = k) = -\left(\frac{5}{6}\right)^3 + EY = -\left(\frac{5}{6}\right)^3 + (3 \times \frac{1}{6}) = -0.0787.$$

2. (a) Let $X_1 = 1$. For $i \geq 2$, let X_i be the number of additional coupons needed to get the ith new type, once we already have $i - 1$ different types. Since there are $m - i + 1$ out of m new types left, $X_i \sim \text{Geom}(\frac{m-i+1}{m})$ for $i \geq 2$. Then $X = \sum_{i=1}^{m} X_i$. **[3]**

   (b) Since $EX_i = \frac{m}{m-i+1}$ (including for $i = 1$ since then both sides equal 1), **[2]**

$$EX = \sum_{i=1}^{m} EX_i = \frac{m}{m} + \frac{m}{m-1} + \cdots + \frac{m}{1} = m\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m}\right).$$

   (This is approximately $m \log m$ for m large.)

3. (a) $P(Y > m) = P(\text{first } m \text{ trials fail}) = (1 - p)^m$. **[2]**

   (b)

$$P(\text{no success by time } t) = P(Y > t/\delta) = (1 - p)^{t/\delta} \quad (\text{power should be } \lfloor t/\delta \rfloor \text{ but this gives same limit})$$
$$= \left((1 - \lambda\delta)^{1/\delta}\right)^t \to (e^{-\lambda})^t = e^{-\lambda t}.$$

   If we set X = time of first success, in the limit we have $P(X > t) = e^{-\lambda t}$ so $F_X(t) = P(X \leq t) = 1 - e^{-\lambda t}$ and $f_X(t) = F_X'(t) = \lambda e^{-\lambda t}$, which is the exponential pdf. **[3]**

4. Let S_0 be the event that 0 was sent, and let F_1 be the event that the message received is 1. By Bayes' Theorem,

$$P(S_0 \mid F_1) = \frac{P(F_1 \mid S_0)P(S_0)}{P(F_1 \mid S_0)P(S_0) + P(F_1 \mid S_0^c)P(S_0^c)}$$
$$= \frac{P(-1 + N \geq .5)\frac{1}{2}}{P(-1 + N \geq .5)\frac{1}{2} + P(2 + N \geq .5)\frac{1}{2}}$$
$$= \frac{P(N \geq 1.5)}{P(N \geq 1.5) + P(N \geq -1.5)}.$$

   Let $Z \sim N(0, 1)$. Since **[6]**

$$P(N \geq 1.5) = P\big(Z \geq \frac{1.5 - 0}{.5}\big) = P(Z \geq 3) = 1 - \Phi(3) = 0.0013,$$
$$P(N \geq -1.5) = P\big(Z \geq \frac{-1.5 - 0}{.5}\big) = P(Z \geq -3) = \Phi(3) = 0.9987,$$

   we obtain

$$P(S_0 \mid F_1) = \frac{.0013}{.0013 + 0.9987} = 0.0013.$$

   (For a shorter answer, note that $P(F_1) = \frac{1}{2}$ by symmetry. The first denominator after application of Bayes' Theorem is simply $P(F_1)$, and this can be cancelled with $P(S_0)$ in the numerator. With this simpler approach, we do not have to compute $P(N \geq -1.5)$.)

5. Jupyter notebook for overbooking problem

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy.stats

N = 525;
p = 0.07;
seats = 500;
numtrials = 40000;

# (a): We want the probability that the number of passengers
# who do not show up is at most (N-seats-1). So:

prob_overbook_actual = scipy.stats.binom.cdf(N-seats-1, N, p)
prob_overbook_actual
```

[1]: 0.014067102732306875

[2]:
```
# (b): Compute the Poisson approximation: the appropriate parameter
# is lambda = N p:

prob_overbook_poisson = scipy.stats.poisson.cdf(N-seats-1, N*p)
prob_overbook_poisson
```

[2]: 0.01688480901099353

[3]:
```
# (c): Simulate the no-shows for the bookings numtrials times. This␣
↪amounts to generating numtrials Bin(N,p) numbers.
no_shows = scipy.stats.binom.rvs(N,p,size=numtrials);

# Calculate the proportion of these bookings which are overbooked:
overbooked = no_shows < (N - seats);


prop_overbook = sum(overbooked)/numtrials
prop_overbook
```

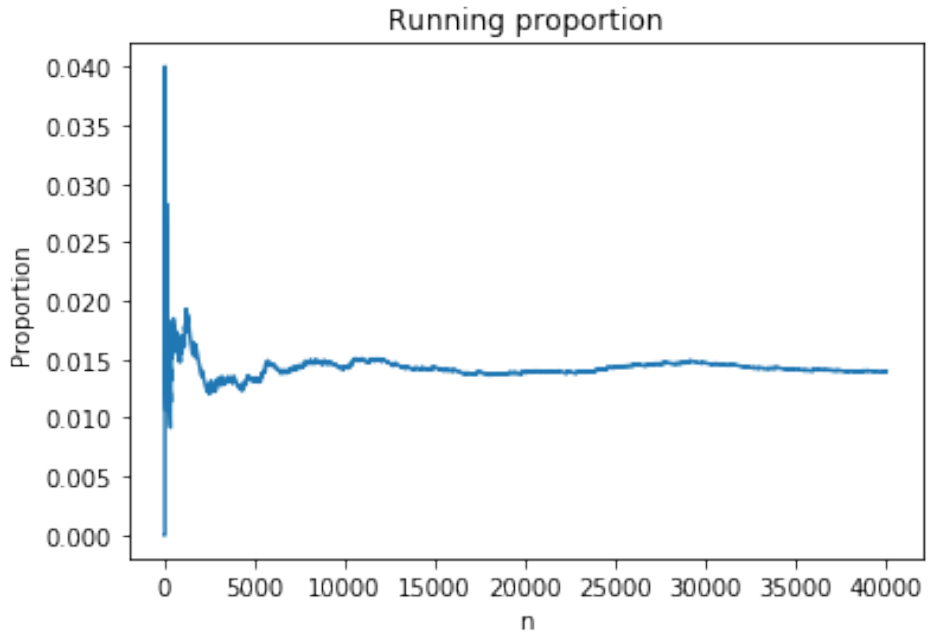
[3]: 0.014025

[4]:
```
#To calculate O_n, recall the cumsum function:

O_n = np.cumsum(overbooked);
running_proportion = O_n / (np.arange(numtrials)+1);
```

[5]:
```
# Now plot:
plt.plot(running_proportion)
plt.xlabel("n")
plt.ylabel("Proportion")
plt.title("Running proportion")
```

[5]: Text(0.5, 1.0, 'Running proportion')


Running proportion
Proportion
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
0
5000
10000
15000
20000
25000
30000
35000
40000
n