January 8, 2020

Math 318 Assignment 1: Due Wednesday, January 15 at start of class

I. Problems to be handed in: Always provide clear explanations of your solutions, not merely answers. In particular, in problems involving permutations and combinations, be sure to explain all factors arising in your solution.

1. A coin is tossed until either two Heads appear successively, or until the fifth toss, whichever comes first. Write down the sample space, and determine the probability of each outcome in the sample space.

2. Ross: Chapter 1 #4.

3. An ecology graduate student goes to a pond and captures 40 water beetles, marks each with a dot of paint, and then releases them. A few days later she goes back and captures another sample of 60, finding 12 marked beetles and 48 unmarked.

   (a) Assuming that the pond contains n beetles, determine the probability $L(n)$ that a catch of 60 beetles will contain 12 marked ones.

   (b) Show that the function $L(n)$ is initially an increasing function of n which then becomes decreasing after reaching a maximum value. Find the *maximum likelihood estimate* for n; that is the value of n which maximizes $L(n)$.
   Hint: when does the inequality $L(n)/L(n-1) \le 1$ hold?

4. (a) Compute the probability that a poker hand contains:

      i. one pair ($aabcd$ with a, b, c, d distinct face values; answer: 0.4226)
      ii. two pairs ($aabbc$ with a, b, c distinct face values; answer: 0.04754)

   (b) Poker dice is played by simultaneously rolling 5 dice. Compute the probabilities of the following outcomes:

      i. one pair ($aabcd$ with a, b, c, d distinct numbers; answer: 0.4630)
      ii. two pairs ($aabbc$ with a, b, c distinct numbers; answer: 0.2315)

5. The number of ways to place n *distinguishable* balls in m urns is m^n, since each ball can be placed in any one of the m urns. The multinomial coefficient $\binom{n}{n_1,\ldots,n_m} = \frac{n!}{n_1!\cdots n_m!}$ counts the number of ways that n_i balls are in urn i for each $i = 1, 2, \ldots, m$, so when each ball is randomly assigned to an urn, the probability that n_i balls are in urn i, for each i, is equal to $\binom{n}{n_1,\ldots,n_m} m^{-n}$. Systems described by these probabilities are said to obey Maxwell–Boltzmann statistics.

   (a) Suppose instead that the balls are *indistinguishable*; now we speak of Bose–Einstein statistics. When there are $m = 2$ urns, the number of ways of putting the n balls in the 2 urns is $n + 1$, because an outcome is specified by saying how many balls are in urn 1 and the possibilities are $\{0, 1, 2, \ldots, n\}$. For the case of general $m \ge 1$, how many ways are there to place n indistinguishable balls in m urns?
   Hint: This is the number of ways to arrange $m - 1$ barriers among a row of n balls, e.g., for $n = 7$ and $m = 3$ the configuration with $n_1 = 0, n_2 = 2, n_3 = 5$ is $|\circ\circ|\circ\circ\circ\circ\circ$.

   (b) Indistinguishable particles are said to obey Fermi–Dirac statistics if all arrangements that have at most one ball per urn have the same probability. How many ways can n of these particles be put into $m \ge n$ urns?

6. Use Python (in a Jupyter notebook) to write a program that will do the following.

   (a) Write a function `birthday(n)` that:
   (i) generates a vector containing n numbers uniformly distributed on $\{1, 2, \ldots, 365\}$ (think of this as the list of birthdays of n people. You can use the randint function from numpy.random.),
   (ii) returns 1 if there is at least one pair of people with coinciding birthdays (a "match"), and 0 otherwise.

   (b) For $n = 2$ to 60, run the function `birthday(n)` 1000 times, and compute the proportion $X(n)$ of the 1000 times in which there was a match. Hint: set `A(n,i)=birthday(n)` for $i = 1, \ldots, 1000$ and put $X(n) = \frac{1}{1000} \sum_{i=1}^{1000} A(n, i)$.

   (c) Let
   $$Y(n) = 1 - \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n}.$$
   On a single graph, plot $X(n)$ and $Y(n)$ vs $n \in [2, 60]$.

   Print and submit your program and output. Your code should be easy to read and adequately commented. Submit hard copies, no email, and do not use other programming languages.

II. Recommended problems: These provide additional practice but are not to be handed in. Starred problems have solutions in the text, and answers are given otherwise.

Ross, Chapter 1: 2*, 5*, 6, 7, 8, 9*, 10, 11 ($\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$), 17*.