Exercises from the textbook (Grinstead & Snell 2nd revised edition)

**Marking:** The first 3 exercises (1. out of 3, 2. out of 5, 3. out of 4), total 12.

1. Section 3.2 (p.115) Ex. 20
   Solution: (a) The number of six-cards hands is \( \binom{52}{6} \), and the number of six-cards hands with hearts only is \( \binom{13}{6} \), hence the probability is
   \[
   \frac{\binom{13}{6}}{\binom{52}{6}} \approx 0.000084.
   \]
   (b) The probability is
   \[
   \frac{\binom{4}{3} \binom{4}{2} \binom{4}{1}}{\binom{52}{6}} \approx 0.0000047.
   \]
   (c) The probability is (where \( \binom{4}{2} \) is the number of ways to choose the two colors for the suites)
   \[
   \frac{\binom{4}{2} \binom{13}{3} \binom{13}{3}}{\binom{52}{6}} \approx 0.024.
   \]

2. Multinomial coefficients
   For \( m \geq 1 \) and \( n_1, \ldots, n_m \) positive integers, with the notation \( n \) for the sum
   \( n_1 + \cdots + n_m \), denote by \( C[n_1, \ldots, n_m] \) the number of ways to place \( n \) balls (of
   labels 1, 2, \ldots, n) into \( m \) urns \( U_1, \ldots, U_m \), such that there are \( n_1 \) balls falling into
   \( U_1, \ldots, n_m \) balls falling into \( U_m \).
   a) Show that for \( m = 2 \), \( C[n_1, n_2] = \binom{n}{n_1} \).
      Solution: the configuration is completely encoded by the subset (among \( n \) balls) of
      \( n_1 \) balls falling in \( U_1 \). Hence \( C[n_1, n_2] = \binom{n}{n_1} \).
   b) For \( m = 3 \), show that
      \[
      C[n_1, n_2, n_3] = \binom{n}{n_1} \binom{n-n_1}{n_2},
      \]
      and show that this simplifies to
      \[
      C[n_1, n_2, n_3] = \frac{n!}{n_1! n_2! n_3!}.
      \]
      Solution: there are \( \binom{n}{n_1} \) possibilities for choosing which balls fall in \( U_1 \). Then,
      among the \( n - n_1 \) remaining balls, there are \( \binom{n-n_1}{n_2} \) possibilities for choosing which
      balls fall in \( U_2 \), and then this determines which balls fall in \( U_3 \) (the \( n - n_1 - n_2 \)
      remaining ones) so that there is no choice left. By the multiplication rule we conclude that
      \[
      C[n_1, n_2, n_3] = \binom{n}{n_1} \binom{n-n_1}{n_2}.
      \]
      This rewrites as follows
      \[
      C[n_1, n_2, n_3] = \binom{n}{n_1} \frac{n!}{n_1!(n-n_1)!} \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}.
      \]
Note that the \((n - n_1)!\) in the numerator and denominator cancel out, and note also that \(n - n_1 - n_2 = n_3\), from which we conclude that \(C[n_1, n_2, n_3] = \frac{n!}{n_1!n_2!n_3!}\).

c) Show that \(C[n_1, n_2, n_3]\) also gives the numbers of words of length \(n\) with letters \(a, b\) or \(c\), such that \(a\) appears \(n_1\) times, \(b\) appears \(n_2\) times, \(c\) appears \(n_3\) times.

Solution: the positions of the \(a\)'s (resp. of the \(b\)'s, resp. of the \(c\)'s) give the subset of balls in \(\{1, \ldots, n\}\) falling in \(U_1\) (resp. in \(U_2\), resp. in \(U_3\)).

d) Compute how many words can be formed from the letters of “assesses”.

Solution: this word has one ‘\(a\)’, two ‘\(e\)’, and five ‘\(s\)’, hence from what precedes, the number of words that can be formed equals \(C[1, 2, 5] = \frac{8!}{3!5!} = 168\).

e) Generalize the approach of (b) to show that

\[
C[n_1, \ldots, n_m] = \frac{n!}{n_1! \cdots n_m!},
\]

Solution: there are \(\binom{n}{n_i}\) possibilities for choosing which balls fall in \(U_1\), then among the \(n - n_1\) remaining balls there are \(\binom{n-n_1}{n_2}\) possibilities for choosing which balls fall in \(U_2\), then among the \(n - n_1 - n_2\) remaining balls there are \(\binom{n-n_1-n_2}{n_3}\) possibilities for choosing which balls fall in \(U_3\), ..., until we have placed \(n_1\) balls in \(U_1\), \(n_2\) balls in \(U_2, \ldots, n_{m-1}\) balls in \(U_{m-1}\), and we place the remaining \(n_m\) balls in \(U_m\). By the multiplication rule we obtain

\[
C[n_1, \ldots, n_m] = \binom{n}{n_1} \binom{n-n_1}{n_2} \cdots \binom{n-(n_1+\cdots+n_{m-2})}{n_{m-1}} = \prod_{i=1}^{m-1} \binom{n-a_i}{n_i},
\]

where we define \(a_1 = 0, a_i = n_1 + \cdots + n_{i-1}\) for \(i \in \{2, \ldots, m\}\). Hence we have

\[
C[n_1, \ldots, n_m] = \prod_{i=1}^{m-1} \frac{(n-a_i)!}{n_i!(n-a_i-n_i)!}.
\]

Now, note that \(a_i + n_i = a_{i+1}\) for every \(i\) in \(\{1, \ldots, m-1\}\), hence

\[
C[n_1, \ldots, n_m] = \prod_{i=1}^{m-1} \frac{(n-a_i)!}{n_i!(n-a_{i+1})!} = \frac{(n-a_1)! \cdots (n-a_{m-1})!}{n_1! \cdots n_{m-1}! (n-a_1) \cdots (n-a_{m-1})!}.
\]

Note that the factors \((n-a_i)!\) cancel out except for \((n-a_1)!\) in the numerator and \((n-a_m)!\) in the denominator; note also that \((n-a_1)! = n!\) (since \(a_1 = 0\)) and \((n-a_m)! = n_m!\) (since \(a_m = n_1 + \cdots + n_{m-1} = n - n_m\)). Overall we obtain

\[
C[n_1, \ldots, n_m] = \frac{n!}{n_1! \cdots n_m!}.
\]

3. Section 4.1 (p.150) Ex. 3

Solution:
(a) If the first outcome is 4, the second outcome needs to be 4, 5 or 6 for the sum to be greater than 7, so the probability is \(3/6 = 1/2\).
(b) The event \(F = \{\text{first outcome greater than 3}\}\) has probability \(1/2\), and we want to find \(P(E|F)\), where \(E\) is the event \{sum greater than 7\}. The elements of \(E \cap F\) are \{4,4\}, \{4,5\}, \{4,6\}, \{5,3\}, \{5,4\}, \{5,5\}, \{5,6\}, \{6,2\}, \{6,3\}, \{6,4\}, \{6,5\}, \{6,6\}\), so that we have \(P(E \cap F) = 12/36 = 1/3\). Hence \(P(E|F) = (1/3)/(1/2) = 2/3\).
(c) If the first outcome is 1, there is no way the second outcome (which is at most 6) can make the sum larger than 7, hence the probability is 0.

(d) We can argue as in (b), with now $F = \{ \text{first outcome less than 5} \}$. Note that $P(F) = 4/6 = 2/3$, and we have $E \cap F = \{(2, 6), (3, 5), (3, 6), (4, 4), (4, 5), (4, 6)\}$, so that $P(E \cap F) = 6/36 = 1/6$. Hence $P(E|F) = (1/6)/(2/3) = 1/4$.

4. Section 4.1 Ex. 9 (assuming each newborn has equal chances to be boy or girl)
Solution: the sample space is $\{BB, BG, GB, GG\}$, each with probability 1/4.
(a) We have $P(BB|\{BB, BG, GB\}) = 1/3$
(b) We have $P(BB|\{BB, BG\}) = 1/2$

5. Section 4.1 Ex. 15
Solution:
(a) let $F = \{ \text{at least one ace} \}$. Then $P(F) = \frac{\binom{4}{1} \binom{48}{1}}{\binom{52}{1}}$. And let $E = \{ \text{exactly two aces} \}$. Then $E \subset F$, we have $P(E \cap F) = P(E)$, so that

$$P(E|F) = \frac{\frac{\binom{4}{2} \cdot \binom{48}{1}}{\binom{52}{2}}}{\frac{\binom{52}{2}}{\binom{52}{1}}} = \frac{2223}{7249} \approx 30.7\%.$$

(b) let $F = \{ \text{has ace of spade} \}$. Then $P(F) = \frac{\binom{13}{1}}{\binom{52}{1}}$. And let $E = \{ \text{exactly two aces} \}$. Beware that $E$ is not included in $F$. However we can directly compute $P(E \cap F) = \frac{\binom{4}{1} \binom{48}{1}}{\binom{52}{2}}$, and we obtain

$$P(E|F) = \frac{\frac{\binom{4}{1} \binom{48}{1}}{\binom{52}{2}}}{\frac{\binom{52}{2}}{\binom{52}{1}}} = \frac{8892}{20825} \approx 42.7\%.$$

6. Section 4.1 Ex. 18
Solution: Let $E$ be the event that the test is positive. We can use Bayes formula to find

$$P(d_1|E) = \frac{P(E|d_1) \cdot P(d_1)}{P(E)} = \frac{P(E|d_1) \cdot P(d_1)}{0.8/3} = \frac{0.8}{0.8 + 0.6 + 0.4} = \frac{4}{9},$$

and similarly $P(d_2|E) = \frac{0.6}{0.8 + 0.6 + 0.4} = \frac{3}{5}$, and $P(d_3|E) = \frac{0.4}{0.8 + 0.6 + 0.4} = \frac{2}{5}$.

7. Section 4.1 Ex. 22
Solution: let $E$ be the event to have heads 6 times in a row, $D$ be the event to use the double-headed coin, and $N = \bar{D}$ the event to use a non-double-headed coin. We have $P(D) = 1/65$ and $P(N) = 64/65$. Note that $P(E|D) = 1$ and $P(E|N) = 1/2^6 = 1/64$. Hence $P(E \cap D) = P(E|D) \cdot P(D) = 1/65$. And $P(E) = P(E|D) \cdot P(D) + P(E|N) \cdot P(N) = \frac{1}{65} + \frac{1}{4 \cdot 65} = \frac{2}{65}$. Hence $P(D|E) = (1/65)/(2/65) = 1/2$.

8. Section 4.1 Ex. 24
Solution: let $i_0 \in \{1, \ldots, n\}$ be the trial on which we condition, let $E$ be the event
of having \(k\) heads in the \(n\) trials, and let \(E'\) be the subset of \(E\) where there is “head” at the position \(i_0\). Then the searched probability is

\[
P(E'|E) = \frac{P(E')}{P(E)} = \frac{|E'|}{|E'|/2^n} = \frac{|E'|}{|E|/2^n} \]

Clearly we have \(|E| = \binom{n}{k}\) (number of words on \(\{H,T\}\) of length \(n\) with \(k\) occurences of \(H\)), and we have \(|E'| = \binom{n-1}{k-1}\) (for any outcome in \(E'\), the word obtained by deleting the \(H\) at position \(i_0\) can be any word of length \(n - 1\) on \(\{H,T\}\) with \(k - 1\) occurences of \(H\)). Hence

\[
P(E'|E) = \binom{n-1}{k-1} / \binom{n}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{n!}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!} \cdot \frac{k!}{n!} = \frac{k}{n}.
\]