1. A bin of 4 transistors has 2 defective ones. You repeatedly take (without replacement) transistors out of the bin and test them. Let \( N_1 \) be the rank (in \([1,4]\)) of the first defective one found and let \( N_2 \) be the rank (in \([1,4]\), with \( N_2 > N_1 \)) of the second defective one found. Compute the joint p.m.f. of \( X = N_1 \) and \( Y = N_2 - N_1 \), and compute the marginal p.m.f.'s for \( X \) and \( Y \).

Solution: The sample space (\( D \) stands for ‘defective’, \( G \) for ‘good’) is the set \( \{DDGG, DGDG, DGGD, GDGD, GDDG, GGDD \} \) where the values for the pair \((X,Y)\) are respectively \((1,1), (1,2), (1,3), (2,2), (2,1), (3,1)\), each of probability \(1/6\). Hence \( p(x,y) = 1/6 \) if \((x,y)\) belongs to \(\{(1,1), (1,2), (1,3), (2,2), (2,1), (3,1)\}\). The marginal p.m.f. for \(X\) is \(p_X(1) = 3/6 = 1/2\), \(p_X(2) = 2/6 = 1/3\) and \(p_X(3) = 1/6\), and the p.m.f. for \(Y\) is the same (since the joint p.m.f. is symmetric in \(x\) and \(y\)).

2. Let \(X, Y\) have joint p.d.f. \(f(x,y) = c(y-x)e^{-y}\) for \(-y \leq x \leq y\) and \(y \geq 0\) and \(f(x,y) = 0\) else.
(a) Find \(c\).
(b) Find the marginal p.d.f.'s \(f_X(x)\) and \(f_Y(y)\).

Solution: (a) We have
\[
\int \int f(x,y) \, dx \, dy = c \int_{-\infty}^{\infty} e^{-y} \left( \int_{-\infty}^{y} (y-x) \, dx \right) \, dy
\]
\[
= c \int_{-\infty}^{\infty} e^{-y} (yx - x^2/2) \, dy
\]
\[
= c \int_{0}^{\infty} e^{-y}2y^2 \, dy = 2c \int_{0}^{\infty} e^{-y}y^2 \, dy = 4c.
\]
Since \( \int \int f(x,y) \, dx \, dy = 1 \) we conclude that \(c = 1/4\).
(b) For \(x \in \mathbb{R}\) we have
\[
f_X(x) = \int f(x,y) \, dy = \frac{1}{4} \int_{|x|}^{\infty} e^{-y}(y-x) \, dy.
\]
Next, we have for every \(a \geq 0\): \(\int_{a}^{\infty} e^{-y} = -e^{-y}\big|_{a}^{\infty} = e^{-a}\), and using integration by parts we have
\[
\int_{a}^{\infty} ye^{-y} \, dy = -ye^{-y}\big|_{a}^{\infty} + \int_{a}^{\infty} e^{-y} \, dy = ae^{-a} + e^{-a} = (1+a)e^{-a},
\]

hence
\[
f_X(x) = \frac{1}{4} \left( (1+|x|)e^{-|x|} - xe^{-|x|} \right).
\]
In other words we have $f_X(x) = \frac{1}{4} e^{-x}$ for $x \geq 0$ and $f_X(x) = \frac{1}{4} (1-2x)e^x$ for $x \leq 0$. For the marginal in $Y$ we have for every $y \geq 0$

$$f_Y(y) = \int f(x,y)dx = \frac{1}{4} \int_{-y}^{y} (y-x)e^{-y}dx$$

$$= \frac{e^{-y}}{4} (y \int_{-y}^{y} dx - \int_{-y}^{y} x dx)$$

$$= \frac{e^{-y}}{4} (2y^2 - 0) = \frac{y^2 e^{-y}}{2}.$$

3. Let $X,Y$ be two independent uniform random variable on $[0,1]$. Find the c.d.f. and then the p.d.f. of $Z = |X-Y|$. How does the distribution of $Z$ compare to the distribution of $\min(X,Y)$?

**Solution:** For $0 \leq a \leq 1$, the area corresponding to $|X-Y| \leq a$ is shown in the figure below:

and it clearly has area $1 - 2*[(1-a)^2/2] = 1 - (1-a)^2 = 2a-a^2$. Hence the density function is $2-2a$ for $a \in [0,1]$ and is 0 else. We thus find the same distribution as for the random variable $Z = \min(X,Y)$ (in both cases the complement of $\{Z \leq a\}$ has area $(1-a)^2$).

4. (a) Prove that, for $X,Y$ random variables and $a, b, c, d$ constants,

$$\text{Cov}(a+bX, c+dY) = bd \text{Cov}(X,Y).$$

(b) Prove that

$$\text{Cov}(X+Y,Z) = \text{Cov}(X,Z) + \text{Cov}(Y,Z).$$

(c) Prove that

$$\text{Cov} \left( \sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \text{Cov}(X_i,Y_j).$$

**Solution:** (a) We have

$$E \left[ (a+bX)(c+dY) \right] = ac + bE[Y] + cE[X] + bdE[XY]$$

and

$$E[a+bX] \cdot E[c+dY] = ac + bE[Y] + cE[X] + bdE[X]E[Y],$$

hence

$$\text{Cov}(a+bX, c+dY) = bd \left( E[XY] - E[X]E[Y] \right) = bd \text{Cov}(X,Y).$$
(b) We have
\[
\]
Similarly we have \(\text{Cov}(Z, X + Y) = \text{Cov}(Z, X) + \text{Cov}(Z, Y)\).

(c) Using the previous question we have
\[
\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n} \text{Cov}(X_i, \sum_{j=1}^{m} Y_j) \\
= \sum_{i=1}^{n} \sum_{j=1}^{m} \text{Cov}(X_i, Y_j).
\]

5. Show that if two random variables \(X\) and \(Y\) have the same variance then \(\text{Cov}(X + Y, X - Y) = 0\).

Solution: We have
\[
= E[X^2 - Y^2] - (E[X] + E[Y])(E[X] - E[Y]) \\
\]
hence if \(V[X] = V[Y]\) then \(\text{Cov}(X + Y, X - Y) = 0\).

6. Let \(X\) be a random variable with \(V[X] > 0\), and for \(a\) and \(b\) two constants let \(Y = aX + b\). Show that the correlation coefficient \(\rho(X, Y) = \frac{\text{Cov}(X,Y)}{\sqrt{V[X]}\sqrt{V[Y]}}\) is equal to 1 for \(a > 0\), is equal to 0 for \(a = 0\) and is equal to \(-1\) for \(a < 0\).

Solution: We have \(V[Y] = a^2V[X]\) hence \(\sqrt{V[X]}\sqrt{V[Y]} = |a|V[X]\). And we have \(\text{Cov}(X, Y) = \text{Cov}(X, aX + b) = a\text{Cov}(X, X) = aV[X]\) using Ex.4(a). Hence \(\rho(X, aX + b) = \frac{aV[X]}{|a|V[X]} = \frac{a}{|a|}\), so that \(\rho(X, aX + b) = 1\) if \(a > 0\) and \(\rho(X, aX + b) = -1\) if \(a < 0\).