Recall our model: \( V(r) = E\left(\left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right)^6\right) \).

We found \( V\left(\frac{R}{2^{1/6}}\right) = 0 \), \( \lim_{r \to 0^+} V(r) = \infty \), \( \lim_{r \to \infty} V(r) = 0 \).

What does this tell us about its graph?

What distinguishes this point?
One answer: the "tangent line" there is "horizontal."
But... what do we mean by tangent line?

[Detour: given a slope $m$ and a point $(x_0, y_0)$ on the line, the line is described by the equation $y - y_0 = m(x - x_0).$]

Exercises: reconcile this with the $y = ax + b$ formula from high school, and write out the equations of a few lines:
Question: How do we describe the tangent line to $y = f(x)$ at a particular point $(x_0, y_0)$?

![Diagram showing the tangent line and the expression for its slope.]

- This has slope
  \[
  \frac{f(x_0 + h) - f(x_0)}{h}
  \]
If \( \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \) exists, we define it to be the slope of the tangent line at \((x_0, y_0)\), and call it the derivative of \(f(x)\) at \(x = x_0\), or \(f'(x_0)\).
Examples: What is $f'(1)$ for:

(a) $f(x) = x^2$
(b) $f(x) = x$
(c) $f(x) = 3$.

This has slope $\frac{f(1+h) - f(1)}{h}$.
\[
\lim_{h \to 0} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0} 1+2h+h^2 - 1 = 2.
\]

The tangent line between \((1,1)\) has slope
\[
\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{1+h-1}{h} = \lim_{h \to 0} \frac{h}{h} = 1.
\]

\[
(x) = f(x)
\]

\[
(2x + f(1+h))^2 - 1
\]
The tangent line between $(1, f(1))$ and $(1+h, f(1+h))$ has slope $\frac{f(1+h) - f(1)}{h}$. 

\[
\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = 0.
\]

Summary: The tangent line to $y = f(x)$ at $x = x_0$ is defined to be the line through $(x_0, f(x_0))$ with slope

\[
f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.
\]
The derivative of a function is itself a function. We call the derivative of \( f(x) \), \( f'(x) \) (if it exists).

We define

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.
\]

Examples. What is \( f'(x) \) for:

(a) \( f(x) = x^2 \),

(b) \( f(x) = x \), and

(c) \( f(x) = 3 \)?
(c) \( f'(x) = x^2 \)
Compare the pictures:

\[ y = f(x) \]

\[ y = f'(x) \]
Exercise: For (b) and (c), find and sketch $f'(x)$.

Example: What is the derivative of $f(x) = \frac{1}{x^2}$?
The calculation:

\[
\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left( \frac{(x+h)^2 - x^2}{x^2(x+h)^2} \right)
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{x^2 + 2xh + h^2 - x^2}{x^2(x+h)^2} \right)
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{2xh + h^2}{x^2(x+h)^2} \right)
\]

\[
= \lim_{h \to 0} \frac{2x + h}{x^2(x+h)^2}
\]

\[
= \frac{2x}{x^2} = \frac{2}{x^2}
\]

What is the graph of \(-\frac{2}{x^2}\)?

\[
\lim_{x \to \infty} \frac{2}{x^2} = 0
\]

\[
\lim_{x \to 0} \frac{2}{x^2} = \infty
\]

We also have no integers and a domain of all numbers but 0.
The graph of $\frac{1}{x^2}$ has a hyperbolic shape.

The graph of the derivative $-\frac{2}{x^3}$ shows a curve that approaches zero as $x$ approaches 0.
Example: \( f(x) = |x| \).

we claim the derivative does not exist at 0.

\[
\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}, \text{ if it exists, is equal to } f'(0).
\]

But
\[
\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}, \quad \text{and} \quad |h| = \begin{cases} h & \text{if } h \geq 0, \\ -h & \text{if } h < 0, \end{cases}
\]

so
\[
\frac{|h|}{h} = \begin{cases} \frac{h}{h} & \text{if } h \geq 0, \\ \frac{-h}{h} & \text{if } h < 0 \end{cases} = \begin{cases} 1 & \text{if } h \geq 0, \\ -1 & \text{if } h < 0. \end{cases}
\]