a and b are coprime if their greatest common divisor is 1.

What is the probability that two integers "picked at random" are coprime? Let's call this $P$.

What is the probability that two integers have gcd 2?

We need: the first integer to be even

- probability $\frac{1}{2}$

the second integer to be even

once "divided out" by 2, the remaining integer to be coprime

$P$

So the overall probability is $\frac{P}{2^2}$. 
The probability that two integers have \( \text{gcd} \) 3 is \( \frac{p}{3^2} \).

\[
1 = P + \frac{p}{2^2} + \frac{p}{3^2} + \frac{p}{4^2} + \ldots = P \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \right).
\]

\[
P = \frac{1}{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots}.
\]

Recall the approximation for \( \sin(x) \):

\[
\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.
\]
This can be extended ad infinitum.

\[ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots \]

\[ \frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \ldots \]

\[ \frac{\sin(x)}{x} \text{ looks like:} \]

It has "roots" at \( \pi, -\pi, 2\pi, -2\pi, \frac{3\pi}{2}, -\frac{3\pi}{2}, \ldots \)

So we can write \[ \frac{\sin(x)}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right) \ldots \]

\[ = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\left(1 - \frac{x^2}{4^2\pi^2}\right) \ldots \]
In fact:

\[
\frac{\sin(x)}{x} = 1 + x^2 \left( \frac{-x^2}{2 \cdot 3 \cdot \pi^2} \right) + x^4 \left( \frac{-x^4}{2^2 \cdot 3^2 \cdot \pi^4} \right) + \ldots
\]


... one of the terms will be: \(1\left(\frac{-x^2}{2 \cdot 3 \cdot \pi^2}\right)\left(\frac{-x^2}{2 \cdot 3 \cdot \pi^2}\right)(1)(1)(1)\ldots = \frac{x^4}{2^2 \cdot 3^2 \cdot \pi^4} \ldots\)


... one of the terms will be \((1)(1)(\frac{-x^2}{2 \cdot 3 \cdot \pi^2})(\frac{-x^2}{3 \cdot 3 \cdot \pi^2})(1)(1)(\frac{-x^2}{2 \cdot 3 \cdot \pi^2})(1)(1)\ldots\)
\[
\frac{\pi^2}{6} = \frac{\pi^2}{6} - 1 + \frac{\pi^2}{4} + \frac{\pi^2}{16} + \cdots + \frac{\pi^2}{2^{n-1}} + \frac{\pi^2}{n^2} = 1 + \frac{\pi^2}{4} + \frac{\pi^2}{16} + \cdots + \frac{\pi^2}{2^{n-1}} + \frac{\pi^2}{n^2} - 1
\]

So \( P = 1 \)

\[
\frac{6}{\pi^2} = \left( \cdots + \frac{\pi^2}{4} + \frac{\pi^2}{16} + \cdots + \frac{\pi^2}{2^{n-1}} + \frac{\pi^2}{n^2} \right) \frac{6}{\pi^2} = \left( \cdots + \frac{\pi^2}{4} + \frac{\pi^2}{16} + \cdots + \frac{\pi^2}{2^{n-1}} + \frac{\pi^2}{n^2} \right) \frac{6}{\pi^2}
\]

We must have:

\[
\frac{\pi^2}{2} = \frac{6}{\pi^2} \left( \cdots + \frac{\pi^2}{4} + \frac{\pi^2}{16} + \cdots + \frac{\pi^2}{2^{n-1}} + \frac{\pi^2}{n^2} \right)
\]

But we already knew \( \pi^2 = 6 \).

\[
\left( \cdots - \frac{\pi^2}{4} + \frac{\pi^2}{16} - \cdots + \frac{\pi^2}{2^{n-1}} - \frac{\pi^2}{n^2} \right) \frac{\pi^2}{2} = \cdots - \frac{\pi^2}{4} + \frac{\pi^2}{16} - \cdots + \frac{\pi^2}{2^{n-1}} - \frac{\pi^2}{n^2} - \frac{\pi^2}{2}
\]

Thus the \( \frac{\pi^2}{2} \) term is cancelled.