ASSIGNMENT 6
Solutions

1. Note that \((\frac{1}{2})^{1/2} = (\frac{1}{3})^{1/4}\). Explain why there are infinitely many pairs of numbers \(a < b\) such that \(a^a = b^b\). (Hint: consider the curve \(y = x^x\).

Let \(f(x) = x^x\). We wish to prove that there exist infinitely many numbers \(a\) and \(b\) such that \(f(a) = f(b)\). We start by observing that \(f'(x) = x^x(\log(x) + 1)\). (This may be shown by implicit differentiation, for example.) \(f'(x) = 0\) when \(x = \frac{1}{e}\), with \(f'(x) < 0\) for \(0 < x < \frac{1}{e}\) and \(f'(x) > 0\) for \(\frac{1}{e} < x < \infty\). We conclude that \(f(x)\) has a local minimum at \(x = \frac{1}{e}\).

\[
\begin{array}{c}
\bullet \quad f(x) \\
\hline
\frac{1}{4} \quad a \quad \frac{1}{e} \quad b \quad \frac{1}{2} \quad x \\
\end{array}
\]

It follows that, for any number \(L\) between \(f(\frac{1}{4})\) and \(f(\frac{1}{e})\), there exists a number \(a\) between \(\frac{1}{4}\) and \(\frac{1}{e}\) and a number \(b\) between \(\frac{1}{e}\) and \(\frac{1}{2}\) such that \(f(a) = L\). (This itself follow from a result on continuous functions known as the Intermediate Value Theorem.) Since there are infinitely many choices of \(L\), there are infinitely many pairs \(a\) and \(b\) such that \(f(a) = f(b)\).

2. The model

\[
g(x) = \frac{bc + ax^d}{c + x^d}
\]

is a saturation kinetics model used to describe the weight gain of chickens, \(g(x)\), with respect to the concentration \(x\) of lysine in their diet. Here \(d < b < a < c\) are positive constants, and the domain is restricted to \(x > 0\).

(a) Determine \(\lim_{x \to 0^+} g(x)\) and \(\lim_{x \to \infty} g(x)\).

(b) Determine where \(g(x)\) is increasing and where it is decreasing.

(c) Determine where \(g(x)\) is concave up and where it is concave down.

(d) Draw a large sketch of the graph of \(g(x)\), indicating all the information determined in the previous parts of this question.

(e) Explain in one or two sentences a limitation of this model. Your answer should refer both to the mathematical features of the model and to the physical phenomenon the model is describing.

(a) We have \(\lim_{x \to 0^+} g(x) = b\) and \(\lim_{x \to \infty} g(x) = a\).
(b) We start by differentiating \( g(x) \):
\[
g'(x) = \frac{adx^{d-1} (c + x^d) - dx^{d-1} (bc + ax^d)}{(c + x^d)^2} = \frac{(a-b)cdx^{d-1}}{(c + x^d)^2}.
\]
This is positive on the domain \( x > 0 \), which means that \( g(x) \) is increasing on \( x > 0 \).

(c) To determine concavity, we take the second derivative of \( g(x) \):
\[
g''(x) = \frac{(a-b)cdx^{d-2} (c(d-1) - (d+1)x^d)}{(c + x^d)^3}.
\]
We have \( g''(x) = 0 \) when \( c(d-1) = (d+1)x^d \); that is, when \( x = \left( \frac{c(d-1)}{d+1} \right)^{1/d} \). \( g''(x) \) is positive — hence \( g(x) \) is concave up — on \( \left( 0, \left( \frac{c(d-1)}{d+1} \right)^{1/d} \right) \); and \( g''(x) \) is negative — hence \( g(x) \) is concave down — on \( \left( \frac{c(d-1)}{d+1} \right)^{1/d}, \infty \).

(d)

![Graph of function g(x) with horizontal asymptote y = a and points b and a labeled on the x-axis and y-axis respectively.]

(e) One limitation of the model is the horizontal asymptote \( y = a \). This fails to account for the fact that, at high levels, lysine is toxic. A more accurate model would “plateau” for some distance at \( y = a \) before falling dramatically, to describe the lack of weight gain experienced by a poisoned chicken.