ASSIGNMENT 4

There are three parts to this assignment. The first two parts are on WeBWorK — the link is available on the course webpage. The third part consists of the questions on this page. You are expected to provide full solutions with complete justifications. You will be graded on the correctness and coherence of your solutions, as well as on their elegance. Your solutions must be typed, with your name and student number at the top of the first page. If your solutions are on multiple pages, the pages must be stapled together.

Your written assignment must be handed at the front of the lecture hall before the start of class on Monday, October 16. The two online assignments will close at 9:00 on Monday, October 9 and 9:00 on Monday, October 16.

1. In this question you will prove the identities

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$$

for $h > 0$, using the picture below and the Squeeze Theorem.

(a) Explain why $\triangle OPA$ has area $\frac{\sin(h)}{2}$.
(b) Determine the area of the sector of the circle described by the angle $h$.
(c) Explain why $\triangle OTA$ has area $\frac{\sin(h)}{2 \cos(h)}$.
(d) Use the relationship between the areas in parts (a), (b) and (c) to explain why $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$.
(e) Use part (d) to explain why $\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$. (Hint: multiply by $\frac{\cos(h) + 1}{\cos(h) + 1}$.)

2. A particular Serfling baseline model

$$y(t) = 300.5 + 2.1t + 97.6\cos\left(\frac{2\pi t}{13} - 2.67\right)$$

is used to predict the number of deaths related to influenza and pneumonia in a certain population at time $t$. Suppose the actual number of deaths related to influenza and pneumonia is graphed over time, a periodic model is fitted to them, and it is determined that the tangent lines at multiple points have slopes in excess of 70. Explain why this might cause you to reevaluate the baseline model or to anticipate a health aberration in the population. Your answer should make direct reference to the mathematical model, and may include calculations.
3. $^{14}\text{C}$ is a radioactive isotope of carbon found at a roughly constant proportion in the atmosphere. Plants acquire the same proportion of $^{14}\text{C}$ through photosynthesis, and animals through direct and indirect ingestion of plants. When a plant or animal dies, the total amount of $^{14}\text{C}$ it contains begins to decrease via radioactive decay. The half-life of $^{14}\text{C}$ is approximately 5730 years. Plant and animal matter can be dated by measuring the amount of $^{14}\text{C}$ it contains, a technique known as radiocarbon dating.

(a) Suppose a wood fragment is found to have 76% of the amount of original $^{14}\text{C}$ anticipated through other archaeological means to have been in the atmosphere at the time of the tree’s death. Determine the fragment’s age.

(b) Suppose a bivalve shell fragment is also found to have 76% of the amount of original $^{14}\text{C}$. In this case, after doing the analysis as in part (a), an archaeologist might subtract around 440 years from the calculated age. This is to account for the fact that deep ocean water mixing with surface water produces a reservoir effect that makes fragments from aquatic organisms seem older than they actually are. Explain the reservoir effect in two or three sentences illustrated by a well-labelled graph. Your graph should be of the decay curve of $^{14}\text{C}$ over time, along with intervals on the $t$-axis describing the “calculated age” and “actual age” of the shell fragment.

4. The equation

$$x^4 = x^2 + y^2, \ (x, y) \neq (0, 0)$$

describes a curve studied by the ancient Greek mathematician Eudoxus.

(a) Use implicit differentiation to find an expression for $\frac{dy}{dx}$.

(b) Find both points where $\frac{dy}{dx}$ is undefined. (Your answer should exclude $(x, y) = (0, 0)$.)

(c) Use implicit differentiation to find an expression for $\frac{dx}{dy}$.

(d) Find both points where $\frac{dx}{dy} = 0$, and describe in plain English what you think the curve looks like at those points.