1. A hemoglobin molecule binds multiple oxygen molecules at distinct sites. Moreover, the binding is co-operative: the more oxygen molecules that bind to the hemoglobin, the higher the binding affinity at the remaining sites. For example, in oxygen-rich environments like lungs, the proportion of sites that are occupied by oxygen molecules is high, which makes the hemoglobin in that environment especially effective at binding even more oxygen. This relationship between the concentration of oxygen, \( x \), and the proportion of occupied binding sites, \( f(x) \), may be modelled by the Hill equation

\[
f(x) = \frac{x^2}{x^2 + k^2},
\]

where \( k \) is nonzero.

(a) Find the horizontal asymptote of \( f(x) \).

(b) Describe in a few sentences the physical interpretation of the horizontal asymptote found in part (a).

(c) Use the limit definition of derivative to calculate \( f'(x) \).

(d) It should be apparent in your answer to part (c) that \( f'(x) > 0 \) for all \( x > 0 \). Describe in a few sentences the physical interpretation of what this means.

(a) We have

\[
\lim_{x \to \infty} \frac{x^2}{x^2 + k^2} = \lim_{x \to \infty} \frac{1}{1 + \frac{k^2}{x^2}} = 1.
\]

Thus \( f(x) \) has a horizontal asymptote \( y = 1 \).

(b) The horizontal asymptote \( y = 1 \) describes the fact that in oxygen-saturated environments (where \( x \) is very large), almost all binding sites are occupied.

(c) We have

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left( \frac{(x+h)^2}{x^2 + k^2} - \frac{x^2}{x^2 + k^2} \right)
= \lim_{h \to 0} \frac{1}{h} \left( (x+h)^2(x^2 + k^2) - x^2((x+h)^2 + k^2) \right)
= \lim_{h \to 0} \frac{k^2(x+h)^2 - k^2x^2}{2k^2(x+h)^2 + k^2x^2}
= \lim_{h \to 0} \frac{(x^2 + k^2)((x+h)^2 + k^2)}{2k^2x + k^2h}
= \frac{x^2 + k^2}{(x^2 + k^2)^2}.
\]

(d) The fact that \( f'(x) > 0 \) for all \( x > 0 \) means that any increase in the concentration of oxygen, no matter how small, will yield an increase in the proportion of occupied binding sites.
2. Use the limit definition of derivative to show that $f(x) = |x^2 - 5x + 4|$ is not differentiable at $x = 1$.

If $f(x)$ were differentiable at $x = 1$, the limit
\[
\lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{f(1 + h)}{h} = \lim_{h \to 0} \frac{|(1 + h)^2 - 5(1 + h) + 4|}{h} = \lim_{h \to 0} \frac{|h^2 - 3h|}{h} = \lim_{h \to 0} \frac{|h(h - 3)|}{h}
\]
would exist. However, when $h$ is small, we have
\[
|h(h - 3)| = |h||h - 3| = |h|(3 - h) = \begin{cases} 
- h(3 - h) & \text{if } h < 0 \\
(3 - h) & \text{if } h \geq 0
\end{cases}
\]
that is,
\[
\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{-h(3 - h)}{h} = \lim_{h \to 0^-} (h - 3) = -3
\]
and
\[
\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{h(3 - h)}{h} = \lim_{h \to 0^+} (3 - h) = 3.
\]
The limit therefore does not exist.