Work

A. Motivation

Previously, we applied the mechanism of the integral — in general, the idea of splitting up a large object into infinitesimally small components — to volumes of solids. Our second application of the mechanism is to the physical concept of work.

B. The definition of work

1. Work is the energy identified with the action of a force. We define the work \( W \) done by a constant force \( F \) on a point that is displaced a distance \( d \) in the direction of the force to be

\[
W = Fd.
\]

It may be measured in newton-metres; that is, joules (J).

2. Example. A person lifting an object off the ground does positive work on the object: he applies an upward force on the object, which is displaced upward. The same person lowering the object back to the ground does negative work: he applies an upward force on the object (against the force of gravity), which is displaced in the opposite direction. The person does no work moving the object at a constant speed horizontally: the force applied is still upward against the force of gravity, but there is no force applied in the direction of displacement.

3. Defining work becomes rather more difficult if the force is not applied constantly.

As in the case of volume, our definition of work parallels our definition of area. Suppose an object moves along the \( t \)-axis from \( t = l \) to \( t = r \), with a force of \( f(t) \) acting in the same direction on the object at any point \( t \).

We attain our first approximation by assuming the force is constant over the interval \([l, r]\):

\[
W \approx f(t^*) (r - l),
\]

where \( f(t^*) \) is a “representative” force.

We can improve our approximation by partitioning \([l, r]\) into multiple subintervals of equal width, and selecting representative areas from each subinterval. For example, with three subintervals, we would have

\[
W \approx \sum_{i=1}^{3} f(t_i^*) \left( \frac{r - l}{3} \right).
\]

As we refine our partition into more and more subintervals, we expect our approximation to get closer and closer to the “actual” work \( W \).
4. **Definition.** Suppose an object moves along the \( t \)-axis from \( t = l \) to \( t = r \), with a force of \( f(t) \) acting in the same direction on the object at any point \( t \). The work done on the object is equal to

\[
W = \int_{t}^{r} f(t) \, dt
\]

provided this integral exists.

5. **Example.** Suppose a tank in the shape of an inverted pyramid of height 11 m and base side length 8 m is filled with water. We wish to find the work required to empty the tank by pumping all the water to the top of the tank.

It is convenient to place the origin at the top of the tank and the \( t \)-axis pointing downward. We consider the water to be divided into small slices of thickness \( \Delta t \).

At \( t \), the cross-sectional area is \( 4 \left(4 - \frac{4}{11} t\right)^2 \) square metres. (To see this, consider the pyramid side-on, and observe that the top edge may be described by the line \( y = 4 - \frac{4}{11} t \).)

Thus the slice has volume \( 4 \left(4 - \frac{4}{11} t\right)^2 \Delta t \), and the work done to raise this slice a distance \( t \) to the top of the tank is equal to \( 4\rho g t \left(4 - \frac{4}{11} t\right)^2 \Delta t \), where \( \rho \) is the density of water (which we take to be equal to 1000 kg/m\(^3\)), and \( g \) is the acceleration due to gravity (which we take to be equal to 9.8 m/s\(^2\) on or near
the surface of the Earth). Summing the work over all slices and taking the limit yields

\[ W = \int_0^{11} 4 \rho g t \left( 4 - \frac{4}{11} t \right)^2 dt \]

\[ = 4 \rho g \int_0^{11} \left( 16t - \frac{32}{11} t^2 + \frac{16}{121} t^3 \right) dt \]

\[ = 4 \rho g \left( 8t^2 - \frac{32}{33} t^3 + \frac{4}{121} t^4 \right) \bigg|_0^{11} \]

\[ \approx 6330720. \]

Thus approximately 6330720 J of work is done.

6. Example. Suppose one end of a 100 m steel rope weighing 91 kg is attached to the lip of the roof of a 120 m-tall building. We wish to calculate the work done in hoisting the rope up to the roof of the building.

It is convenient to place the origin at the top of the building and the \( t \)-axis pointing downward. We consider the rope to be divided into small sections of length \( \Delta t \). At \( t \), the section has length \( \Delta t \) and linear density \( \frac{91}{100} \) kg/m. Thus the work done to raise this section a distance \( t \) to the roof of building is equal to \( \frac{91}{100} gt \Delta t \). Summing the work over all sections and taking the limit yields

\[ W = \int_0^{100} \frac{91}{100} gt \ dt = \frac{91}{200} g t^2 \bigg|_0^{100} = \frac{9100g}{2} = 44590. \]

Thus approximately 44590 J of work is done.

7. The main challenge in the work problems presented here and in the following section is to set up the problem in a useful way; the mathematics is straightforward.

C. Exercises

1. Consider an inverted right circular conical tank of radius 1 m and height 3 m filled with water. Calculate the work done in pumping all the water out the top of the tank.

2. Consider a spherical tank of radius 1 m filled with fluid of density 1240 kg/m\(^3\). Calculate the work done in pumping all the fluid out the top of the tank.
3. Consider three tanks of equal height and volume, filled with water. The first tank is in the shape of a right circular cone. The second tank is the same shape, but inverted. The third tank is cylindrical. In all three cases, all the water is pumped out the top of the tank. In which case is the most work done? In which case is the least work done?

4. Suppose one end of a 100 m steel rope weighing 91 kg is attached to the lip of the roof of a 40 m-tall building. Calculate the work done in hoisting the other end of the rope up to the roof of the building.

5. (a) A 1 kg bucket on a rope of linear density 0.5 kg/m is drawn up a height of 40 m. Calculate the work done.

(b) Calculate the work done in the previous scenario if the bucket leaks at a constant rate so that it is only half full by the time it reaches the top.

6. Hooke’s Law states that the force required to hold a spring stretched or compressed a distance \(d\) beyond its natural length is equal to \(kd\), where \(k\) is a positive constant associated with the spring. Suppose a spring has a natural length of 10 cm, and that a force of 4 N maintains it stretched to a length of 12 cm. Calculate the work done to stretch the spring from 10 cm to 15 cm.

7. Suppose 3 J of work is done to compress the length of a spring by a factor of 2. Calculate the work done to compress the spring by a factor of 3.

8. The Earth exerts a gravitational force on an object of mass \(m\) (in kilograms) a height \(h\) (in metres) above the surface of the Earth of \(
\frac{km}{r^2+h} \)

, where \(k\) and \(r\) are positive constants (\(r\) is the radius of the Earth). Calculate the work done to raise a 1 kg object to a height of 10000 m.