Volumes

A. Motivation

Given a shape, how do we find its area? We saw at the beginning of this course that it is first necessary to define what we mean by “area”. In our case, we used the intuitive notion of areas of rectangles to build up a formal notion of integrals under curves.

The idea of splitting up something large and difficult to understand into things which are infinitesimally small and, apart from their infinitesimal smallness, easy to understand is widely applicable elsewhere. (In fact, it is an idea that predates by centuries the formal calculus of Newton, Leibniz and later Weierstrass. Their crucial addition was rigour.) We shall consider two physical applications of the idea, to volumes and to work.

B. The definition of volume

1. Our definition of volume parallels our definition of area. Consider a three-dimensional solid on the interval \([l, r]\), as pictured below.

   ![](image)

   Our first approximation is to estimate the volume \(V\) to be equal to the area \(A(t^*)\) of a “representative cross section” multiplied by the length \(r - l\) — that is, to the volume of the cylinder of base area \(A(t^*)\) and length \(r - l\).

   ![](image)

   Thus

   \[ V \approx A(t^*)(r - l). \]
We can improve our approximation by partitioning \([l, r]\) into multiple subintervals of equal width, and selecting representative areas from each subinterval. For example, with three subintervals, we would have
\[
V \approx \sum_{i=1}^{3} A(t_i^*) \left( \frac{r - l}{3} \right).
\]

As we refine our partition into more and more subintervals, we expect our approximation to get closer and closer to the “actual” volume \(V\).

2. **Definition.** The volume of a solid on the interval \([l, r]\) having cross-sectional area \(A(t)\) at position \(t\) is equal to
\[
V = \int_{l}^{r} A(t) \, dt
\]
provided this integral exists.

3. **Example.** The volume of a square-based pyramid of height \(h\) and base side length \(b\) is \(\frac{1}{3} b^2 h\).

It is convenient to place the axes relative to the pyramid as follows.

We observe that the cross sections in that case are squares.
Indeed, note that, side-on, the top edge of the pyramid may be described by the line \( y = \frac{b}{2h} t \). The cross-sectional area at \( t \) is therefore \( \frac{b^2}{h^2} t^2 \), and the volume is

\[
V = \int_0^h \frac{b^2}{h^2} t^2 \, dt = \frac{b^2}{h^2} \frac{t^3}{3} \bigg|_0^h = \frac{1}{3} b^2 h.
\]

Note that, if our definition of volume is to be useful, we should be able to recover this answer by taking “slices” of the pyramid along any axis. (In most cases there will be a small number of “natural” axis choices.) We leave it as an exercise to calculate the volume of the pyramid using another axis choice.

C. Volumes by rotation and cylindrical shells

1. Example. A right circular cone of radius \( r \) and height \( h \) has volume \( \frac{1}{3} \pi r^2 h \).

As above, it is convenient to place the axes relative to the cone as follows.

The cross-sectional area at \( t \) is \( \pi \left( \frac{r}{h} t \right)^2 \), and the volume is

\[
V = \int_0^h \pi \left( \frac{r}{h} t \right)^2 \, dt = \pi \frac{r^2}{h^2} \frac{t^3}{3} \bigg|_0^h = \frac{1}{3} \pi r^2 h.
\]
2. **Calculating volumes by rotation.** The example above is an instance of calculating volumes by rotation; that is, the cross-sectional areas are circles. In general, if the region bounded by \( t = l, t = r \) on the left and right, and \( y = 0 \) and \( y = f(t) \) on the bottom and top, is rotated about the \( t \)-axis, the resulting solid has a cross-sectional area at \( t \) of \( \pi f(t)^2 \), and a volume of

\[
V = \int_l^r \pi f(t)^2 \, dt.
\]

3. **Example.** We consider again the right circular cone of radius \( r \) and height \( h \). However, this time we divide the solid into thin “cylindrical shells” rather than slices.

![Diagram of a cone divided into cylindrical shells](image)

At \( t \), the shell has radius \( t \), height \( h - \frac{h}{r} t \), and thickness \( \Delta t \), hence volume \( 2\pi t \left(h - \frac{h}{r} t\right) \Delta t \). Summing the volumes of these shells and taking the limit yields

\[
V = \int_0^r 2\pi t \left(h - \frac{h}{r} t\right) \, dt = \pi h t^2 - \frac{2h}{3r} \pi t^3 \bigg|_0^r = \frac{1}{3} \pi r^2 h.
\]

4. **Calculating volumes by cylindrical shells.** The example above is an instance of calculating volumes by cylindrical shells. In general, if the region bounded by \( t = l, t = r \) on the left and right, and \( y = 0 \) and \( y = f(t) \) on the bottom and top, is rotated about the \( y \)-axis, the resulting solid has a cylindrical shell at \( t \) of area \( 2\pi tf(t) \), and a volume of

\[
V = \int_l^r 2\pi tf(t) \, dt.
\]

5. Note that it is not *a priori* obvious that calculating volumes by rotation and by cylindrical shells yields the same answer in all cases. In one of the questions below, we prove that the methods are consistent given certain highly constrained conditions. Rather more work is necessary to prove that they are consistent in general.

**D. Exercises**

1. Let \( R \) denote the finite region enclosed by \( y = t \) and \( y = t^2 \). Calculate the volume of the solid obtained by rotating \( R \) about the following lines.
   
   (a) The \( t \)-axis.
   
   (b) The \( y \)-axis.
   
   (c) The line \( t = 2 \).
(d) The line \( y = t \). (Hint: write the volume as \( \int_0^{\sqrt{2}} \pi r(z)^2 \, dz \) for a suitable \( r(z) \).)

2. Calculate the volume of a sphere of radius \( r \).

3. Calculate the volume of a tetrahedron of side length \( a \).

4. Let \( R \) denote the region enclosed by the \( t \)-axis and \( y = \frac{2\sqrt{t}}{t+1} \) in the half-plane \( t \geq 0 \). Find the volume of the solid obtained by rotating \( R \) about the \( t \)-axis.

5. Let \( R \) denote the region enclosed by \( y = t^{12} - t^8 + 3t^6 + t^4 + t^3 + 16t - 4 \) and \( y = t^{12} - t^8 + 3t^6 + t^4 + 6t^2 + 8t - 4 \) from \( t = 0 \) to \( t = 4 \). Find the volume of the solid obtained by rotating \( R \) about the \( y \)-axis.

6. Let \( R \) denote the region enclosed by \( y = \sin(t) \) and \( y = \cos(t) \) from \( t = -\frac{\pi}{4} \) to \( t = \frac{3\pi}{4} \). Find the volume of the solid obtained by rotating \( R \) about \( x = -\frac{\pi}{4} \).

7. Let \( R \) denote the region enclosed by \( y = t^3 \) and \( y = 1 \) from \( t = 0 \) to \( t = 1 \). Let \( V \) denote the volume of the solid obtained by rotating \( R \) about the line \( y = 1 \). Find at least one other line on the plane such that the volume of the solid obtained by rotating \( R \) about that line is equal to \( V \).

8. Let \( f(t) \) be a continuously differentiable function (recall that a function is \emph{continuously differentiable} if its derivative is continuous) which passes through the origin and is strictly increasing. Let \( R \) be the region enclosed by the \( t \)-axis and \( y = f(t) \) from \( t = 0 \) to \( t = a \). Let \( S \) denote the solid obtained by rotating \( R \) about the \( y \)-axis. Prove that the volume of \( S \) is equal regardless of whether it is calculated by rotation or by cylindrical shells.