1. Solve the initial value problem
\[ \frac{dy}{dt} - y = 2e^t + 14e^{3t} \]
with the initial condition \( y(0) = 3 \).

The differential equation is linear, so we multiply both sides by the integrating factor \( e^{\int (-1) \, dt} = e^{-t} \) —
\[ \frac{dy}{dt} e^{-t} - ye^{-t} = 2e^t + 14e^{3t} \]
— and then antidifferentiate, getting
\[ ye^{-t} = 2t + 7e^{2t} + C \]
for some constant \( C \). To solve for \( C \), we apply the initial condition \( y(0) = 3 \) to get
\[ C = -4 \], whence
\[ y = 2te^t + 7e^{3t} - 4e^t. \]

2. The Bessel equation is a differential equation which arises in problems of wave propagation. It is given by
\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - 1)y = 0 \] (1)

In this question, you will find one nontrivial solution to the Bessel equation using power series.

(a) Let \( y = \sum_{n \geq 0} a_n x^{\mu+n} \) be a solution to the differential equation. Substitute this into the left-hand side of the Bessel equation, and simplify into a single power series. (Your answer should use the summation notation "\( \sum \), but you may find it helpful to write the first two terms separately.)

(b) Let \( a_0 = 1 \). Find a pattern to describe \( a_1, a_2, a_3, a_4, \ldots \).
(c) Prove that the series \( y \), with the coefficients as described in part (b), converges for all \( x \).

We have
\[ \frac{dy}{dx} = \sum_{n \geq 0} (\mu + n) a_n x^{\mu+n-1} \quad \text{and} \quad \frac{d^2 y}{dx^2} = \sum_{n \geq 0} (\mu + n) (\mu + n - 1) a_n x^{\mu+n-2}. \]

The left-hand side of (1) may be rewritten
\[ \sum_{n \geq 0} \left( (\mu + n) (\mu + n - 1) a_n x^{\mu+n} + (\mu + n) a_n x^{\mu+n+2} - a_n x^{\mu+n} \right) \]
\[ = \sum_{n \geq 0} \left( (\mu + n) (\mu + n - 1) a_n x^{\mu+n} + (\mu + n) a_n x^{\mu+n+2} \right) \]
\[ = \sum_{n \geq 0} \left( (\mu + n)^2 - 1 \right) a_n x^{\mu+n} + a_n x^{\mu+n+2} \).

We divide both sides of (1) by \( x^n \) to get
\[ 0 = \sum_{n \geq 0} \left( (\mu + n)^2 - 1 \right) a_n x^n + a_n x^{n+2} \]
\[ = (\mu^2 - 1) a_0 + (\mu + 1)^2 - 1) a_1 x + \sum_{n \geq 2} \left( (\mu + n)^2 - 1 \right) a_n a_{n-2} x^n. \] (2)
Since this must be identically 0 — that is, all of the coefficients must vanish — and since \( a_0 = 1 \), we must have, from the first two terms,
\[
\mu^2 - 1 = 0 \tag{3}
\]
and
\[
((\mu + 1)^2 - 1) a_1 = 0. \tag{4}
\]
From (3) we take \( \mu = 1 \), which means by (4) that \( a_1 = 0 \). From the remaining terms of (2), we get the recurrence relation
\[
a_n = -\frac{a_{n-2}}{((\mu + n)^2 - 1)} \tag{5}
\]
for \( n \geq 2 \). Thus \( a_3 = a_5 = a_7 = \cdots = 0 \), and
\[
a_2 = -\frac{1}{8}, a_4 = \frac{1}{8 \cdot 24}, a_6 = -\frac{1}{8 \cdot 24 \cdot 48}, a_8 = \frac{1}{8 \cdot 24 \cdot 48 \cdot 80}, \cdots
\]
This may be written as
\[
a_{2n} = \frac{(-1)^n}{2^{2n}(n+1)!n!}.
\]
Thus one solution to (1) is
\[
y = \sum_{n \geq 0} \frac{(-1)^n}{2^{2n}(n+1)!n!} x^{2n+1}.
\]
This converges everywhere, by the Ratio Test.

3. Periodically, you will be asked to post mathematical reflections on a blog. This term, the reflections are to be done in groups of 3-4. Your group must remain the same throughout all assignments, and the names of your group members should be written on each blog post.

The task for your last post is as follows. Look back on your work in MATH 100 and MATH 101 and give advice to the next group of Vantage Science students who take this course. Your advice must involve all of your group members, and it must be creative. It could take the form of a video, a comic strip, a series of photographs, or something else. You will be graded on the creativity of your submission as well as on the soundness of your advice.

Post a link to your submission (one post per group) on Piazza. On your post, as well as on your assignment submission, write down the URL of the blog where your post is located, along with the full names and student numbers of every member in your group.