ASSIGNMENT 4
Solutions

1. Evaluate the integral \( \int_0^r 2e^t \sin(r - t) \, dt \).

Our first move is to factor out the 2:
\[
\int_0^r 2e^t \sin(r - t) \, dt = 2 \int_0^r e^t \sin(r - t) \, dt
\]
We use integration by parts, setting \( u = \sin(r - t) \) and \( dv = e^t \, dt \) (the reverse is also a legitimate choice), whence \( du = -\cos(r - t) \, dt \), \( v = e^t \) and
\[
2 \int_0^r e^t \sin(r - t) \, dt = 2 \left[ e^t \sin(r - t) \bigg|_0^r \right] + 2 \int_0^r e^t \cos(r - t) \, dt = -2 \sin(r) + 2 \int_0^r e^t \cos(r - t) \, dt.
\]
Using integration by parts again on the second term on the right-hand side, we set \( u = \cos(r - t) \) and \( dv = e^t \, dt \), whence \( du = -\sin(r - t) \, dt \), \( v = e^t \) and
\[
2 \int_0^r e^t \sin(r - t) \, dt = -2 \sin(r) + 2e^r - 2\cos(r) - 2 \int_0^r e^t \sin(r - t) \, dt.
\]
This may be rearranged to give
\[
2 \int_0^r e^t \sin(r - t) \, dt = e^r - \sin(r) - \cos(r).
\]

2. The integral \( I_n = \int_0^1 t^n e^t \, dt \) may be calculated using the method of parts repeatedly. Alternately, we may establish a reduction formula which allows us to calculate the integral recursively.

(a) Prove that \( I_n = e - nI_{n-1} \).

(b) Use the reduction formula in part (a) to calculate \( \int_0^1 t^5 e^t \, dt \).

We use integration by parts on \( I_n = \int_0^1 t^n e^t \, dt \), setting \( u = t^n \) and \( dv = e^t \, dt \), whence \( du = nt^{n-1} \, dt \), \( v = e^t \) and
\[
I_n = t^n e^t \bigg|_0^1 - n \int_0^1 t^{n-1}e^t \, dt = e - nI_{n-1}.
\]
This proves part (a). For part (b), we are asked to calculate \( I_5 \). By the reduction formula, we have
\[
I_5 = e - 5I_4
    = e - 5(e - 4I_3)
    = e - 5(e - 4(e - 3I_2))
    = e - 5(e - 4(e - 3(e - 2I_1)))
    = e - 5(e - 4(e - 3(e - 2(e - I_0))))
\]
3. Let \( f(t) \) be an infinitely differentiable function on an interval containing \( a \). Then for any \( x \) in that interval, Taylor’s Theorem (with integral remainder) states that

\[
f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt. \tag{1}
\]

In this question, you will prove this theorem.

(a) Explain why the equation (1) is true in the case \( n = 0 \). (Recall that we define 0! = 1.)

(b) Suppose (1) is true in the case \( n = k \). Prove that it is true in the case \( n = k + 1 \).

(c) Explain in one or two sentences why parts (a) and (b) imply that Taylor’s Theorem is true.

By the Fundamental Theorem of Calculus,

\[
f(a) + \frac{1}{0!} \int_a^x f^{(0+1)}(t)(x-t)^0 \, dt = f(a) + \int_a^x f'(t) \, dt = f(a) + f(x) - f(a) = f(x).
\]

This proves part (a).

Now suppose (1) is true in the case \( n = k \), and we have

\[
f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{k!} \int_a^x f^{(k+1)}(t)(x-t)^k \, dt. \tag{2}
\]

We use integration by parts on the last term on the right-hand side, taking \( u = f^{(k+1)}(t) \) and \( dv = (x-t)^k \, dt \), whence \( du = f^{(k+2)}(t) \, dt \), \( v = \frac{(x-t)^{k+1}}{k+1} \), and

\[
\frac{1}{k!} \int_a^x f^{(k+1)}(t)(x-t)^k \, dt = \frac{1}{k!} \left( \frac{(x-t)^{k+1}}{k+1} \right)_{a}^{x} - \int_a^x \frac{(x-t)^{k+1}}{k+1} f^{(k+2)}(t) \, dt = \frac{1}{k!} \left( \frac{f^{(k+1)}(a)}{k+1} (x-a)^{k+1} + \frac{1}{k+1} \int_a^x f^{(k+2)}(t)(x-t)^{k+1} \, dt \right) = \frac{f^{(k+1)}(a)}{(k+1)!} (x-a)^{k+1} + \frac{1}{(k+1)!} \int_a^x f^{(k+2)}(t)(x-t)^{k+1} \, dt.
\]

Substituting this back into (2) shows that (1) is true in the case \( n = k + 1 \). This proves part (b).

For part (c), we observe that Taylor’s Theorem is a formula — namely, (1) — that we hold is true for all nonnegative integers \( n \). By part (a), the formula is true for the smallest nonnegative integer \( n = 0 \). By part (b), since it is true for \( n = 0 \), it is true for \( n = 1 \). Again by part (b), since it is true for \( n = 1 \), it is true for \( n = 2 \). This chain of logical implication extends too all nonnegative integers.