ASSIGNMENT 2
Solutions

1. Determine where the curve \( y = \int_0^x \frac{7}{t^2 + 8t + 18} \, dt \) is concave down.

By the Fundamental Theorem of Calculus, \( \frac{d}{dx} \int_0^x \frac{7}{t^2 + 8t + 18} \, dt = \frac{7}{x^2 + 8x + 18} \). The curve is concave down where the second derivative \( \frac{d}{dx} \frac{7}{x^2 + 8x + 18} = \frac{-7(2x + 8)}{(x^2 + 8x + 18)^2} \) is negative — namely, when \( x > -4 \).

In the next two problems, you will prove that \( \pi \) is irrational. The proof is by contradiction. Assume that \( \pi = \frac{p}{q} \) where \( p \) and \( q \) are positive coprime integers. Define

\[ f(t) = \frac{t^n(p - qt)^n}{n!} \]

and

\[ F(t) = f(t) - f''(t) + f^{(4)}(t) - f^{(6)}(t) + \cdots + (-1)^n f^{(2n)}(t). \]

You may use the following two facts without proof (though proofs are accessible to you).

**Fact 1.** \( F(0) \) is a positive integer.
**Fact 2.** \( F(\pi) \) is a positive integer.

2. (a) Prove that \( \int_0^\pi f(t) \sin(t) \, dt \leq \frac{\pi^{n+1}p^n}{n!} \).

(b) Prove that \( \frac{d}{dt} (F'(t) \sin(t) - F(t) \cos(t)) = f(t) \sin(t) \).

For \( t \) in \([0, \pi]\), we have \( \sin(t) \leq 1, t \leq \pi \) and \( p - qt \leq p \). Thus

\[ \int_0^\pi f(t) \sin(t) \, dt \leq \int_0^\pi f(t) \, dt \leq \int_0^\pi \frac{\pi^n p^n}{n!} \, dt = \frac{\pi^{n+1} p^n}{n!}. \]

This proves part (a). To prove part (b), we simply apply the Product Rule:

\[ \frac{d}{dt} (F'(t) \sin(t) - F(t) \cos(t)) = F''(t) \sin(t) + F'(t) \cos(t) - (F'(t) \cos(t) - F(t) \sin(t)) = (F''(t) + F(t)) \sin(t). \]

Now

\[ F''(t) = f''(t) - f^{(4)}(t) + f^{(6)}(t) - f^{(8)}(t) + \cdots + (-1)^{n-1} f^{(2n-1)}(t) + (-1)^n f^{(2n+2)}(t) = f(t) - F(t) + (-1)^n f^{(2n+2)}(t). \]

However, since \( f(t) \) is a polynomial of degree \( 2n \), \( f^{(2n+2)}(t) = 0 \). Thus \( F''(t) = f(t) - F(t) \), whence

\[ \frac{d}{dt} (F'(t) \sin(t) - F(t) \cos(t)) = f(t) \sin(t). \]
3. Explain why it follows that \( \pi \) is irrational. (Hint: evaluate \( \int_0^\pi f(t) \sin(t) \, dt \).)

By the Fundamental Theorem of Calculus,

\[
\int_0^\pi f(t) \sin(t) \, dt = (F'(t) \sin(t) - F(t) \cos(t))|_0^\pi = F(\pi) + F(0),
\]

which is a positive integer. However, we showed in question 2 that

\[
\int_0^\pi f(t) \sin(t) \, dt \leq \frac{\pi^{n+1} p^n}{n!}.
\]

Note that there is no restriction on \( n \). In particular, by taking \( n \) to be sufficiently large, it follows that

\[
\int_0^\pi f(t) \sin(t) \, dt < 1,
\]

and is therefore not a positive integer, a contradiction. Therefore \( \pi \) is not rational.