Name (print):

Student number:

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University of British Columbia

MATH 101 (Vantage College): Midterm test

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Date: February 12, 2015
Time: 6:00 p.m. to 7:30 p.m.
Number of pages: 8 (including cover page)
Exam type: Closed book
Aids: No calculators or other electronic aids

Rules governing formal examinations:

Each candidate must be prepared to produce, upon request, a UBC card for identification.

No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:

- Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
- Speaking or communicating with other candidates;
- Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.

Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

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Note that your answers must be in “calculator-ready” form, but they do not have to be simplified.

In general, you may use any result proven in class or on assignments. You may find the following trigonometric identities helpful.

\[
\begin{align*}
\sin^2(x) + \cos^2(x) &= 1 \\
\sin(2x) &= 2 \sin(x) \cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x)
\end{align*}
\]

This page may be used for rough work. It will not be marked.
1. Evaluate the following integrals.

(a) [3 marks] \[ \int_{0}^{\log(2)} e^{t}(1 - e^{t})(1 + e^{t})^{3} \, dt. \]

(b) [3 marks] \[ \int_{a}^{b} \frac{1}{y(1 - y)} \, dy, \text{ where } 0 < a < b < 1. \]
(c) [4 marks] \[ \int_{0}^{\pi/4} \sin(t) \cos(t) \cos(2t) \, dt. \]

(d) [4 marks] \[ \int_{1}^{\infty} \frac{\log(t)}{t^2} \, dt. \]
2. [5 marks] Let $x_1, x_2, \ldots, x_m$ be numbers in the interval $[0, 1]$, and let

$$f(x) = \begin{cases} 0 & \text{if } x = x_1, x_2, \ldots, x_m \\ 1 & \text{otherwise} \end{cases}$$

(In other words, $f(x)$ takes on the value 1 everywhere except at finitely many points.) Prove that $f(x)$ is integrable on $[0, 1]$. 

3. (a) [1 mark] Find an antiderivative of \( \frac{f'(x)}{f(x)} \).

(b) [4 marks] Find a differentiable function \( f(x) \) satisfying the integral equation

\[
f(x) = 2 + \int_{0}^{x^2} \frac{f(\sqrt{t})}{2(t + 1)} \, dt.
\]

for all \( x \geq 0 \). (Hint: use part (a).)
4. [5 marks] The ellipse described by the equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), where \( a \) and \( b \) are positive constants, is pictured below.

Prove that the area of this ellipse is equal to \( \pi ab \).
5. [6 marks] Let $a$ be a positive constant, and $R$ be the region under the curve $y = x^a$ from $x = 0$ to $x = 1$, as pictured below.

Let $V_x$ be the volume of the solid obtained by rotating $R$ about the $x$-axis, and $V_y$ be the volume of the solid obtained by rotating $R$ about the $y$-axis. Prove that there exists no positive constant $a$ such that $V_x = V_y$. 