Name (underline your surname):

Student number:

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University of British Columbia
MATH 100 (Vantage): Midterm test

Date: October 18, 2017
Time: 6:00 p.m. to 7:30 p.m.
Number of pages: 11 (including cover page)
Exam type: Closed book
Aids: No calculators or other electronic aids

Rules governing formal examinations:

Each candidate must be prepared to produce, upon request, a UBC card for identification.

No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:

- Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
- Speaking or communicating with other candidates;
- Purposefully exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.

Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

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For examiners’ use only

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Note that your answers must be in “calculator-ready” form, but they do not have to be simplified.

In general, you may use any result proven in class or on assignments. You do not have to use the precise definition of limit unless you are asked to.

This page may be used for rough work. It will not be marked.
1. Let \( f(x) = \begin{cases} 
3 & \text{if } x \text{ is a rational number} \\
x + 2 & \text{if } x \text{ is an irrational number} 
\end{cases} \).

(a) [3 marks] Prove using the \( \delta - \varepsilon \) definition of limit that \( \lim_{x \to 0} f(x) \neq 2 \). (Hint: draw a picture.)

(b) [2 marks] Prove using the \( \delta - \varepsilon \) definition of limit that \( \lim_{x \to 0} f(x) \) does not exist.
2. (a) [2 marks] Define what it means for a sequence \( \{a_n\} \) to converge to 0, and illustrate your definition with a picture.

(b) [3 marks] A sequence \( \{a_n\} \) is said to be Cauchy if the terms are arbitrarily close to each other provided \( n \) is large enough. Formally, \( \{a_n\} \) is Cauchy if for any \( \varepsilon > 0 \), there exists \( N \) such that \( |a_m - a_n| < \varepsilon \) provided \( m, n \geq N \). Prove that if \( \{a_n\} \) converges to 0, then \( \{a_n\} \) is Cauchy.
3. Determine if the following series are convergent or divergent.

(a) [2 marks] \[ \sum_{n \geq 1} \frac{7}{n3^n} \].

(b) [2 marks] \[ \sum_{n \geq 1} \left( \frac{1}{5} \right)^{1/n} \].
(c) [3 marks] \( \sum_{n \geq 1} \frac{(2n)^n}{n^{2n}}. \)
4. Determine if the following series are conditionally convergent, absolutely convergent, or divergent.

(a) [3 marks] \[ \sum_{n \geq 1} \frac{\cos(\pi n)}{n + \sqrt{n} - 1}. \]

(b) [4 marks] \[ \sum_{n \geq 1} a_n, \text{ where } a_n = \begin{cases} (-1)^n \frac{n^2}{\sqrt{n} + n} & \text{if } n \text{ is prime} \\ \frac{n^2}{\sqrt{n} + n} & \text{if } n \text{ is not prime} \end{cases}. \]
5. (a) [3 marks] Let $\sum_{n \geq 1} a_n$ and $\sum_{n \geq 1} b_n$ be convergent series with all positive terms. Prove that $\sum_{n \geq 1} a_n b_n$ converges.

(b) [1 mark] Write down an example of convergent series $\sum_{n \geq 1} a_n$ and $\sum_{n \geq 1} b_n$ such that $\sum_{n \geq 1} a_n b_n$ diverges.
6. [4 marks] Consider the area illustrated below. Prove that there exists a line through the origin that divides the area exactly in half.
7. [4 marks] Let $f(x)$ be a differentiable function with $f(2) = 3$ and $f'(2) = \pi$. Calculate the limit 

$$
\lim_{t \to 0} \frac{f(2(1 + t)) - 3}{5tf(2(1 + t))}.
$$
8. [1 bonus mark] What character from fiction (e.g. a cartoon character or a superhero) did you feel like during this midterm? Sketch the scene of you battling the midterm.