Rules governing formal examinations:

Each candidate must be prepared to produce, upon request, a UBC card for identification.

No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:

- Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
- Speaking or communicating with other candidates;
- Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.

Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.
Note that your answers must be in “calculator-ready” form, but they do not have to be simplified.

In general, you may use any result proven in class or on assignments. You do not have to use the precise definition of limit. Other than in question 6, you do not have to use the definition of derivative.

This page may be used for rough work. It will not be marked.
1. You must receive a passing grade on this question in order to pass the course. However, if you do not receive a passing grade on this question, you will be given more chances at future dates to rewrite it, even though your original grade will remain.

(a) [1 mark] Consider the function $f(x)$ whose graph is pictured above. Fill in the blank so that the following is a correct proof that $\lim_{x \to 0} f(x) = 0$.

Let $\varepsilon > 0$ be given, and let $\varepsilon < 3$. Can we guarantee $f(x)$ to be within $\varepsilon$ of 0 provided $x$ is sufficiently close to 0? We can. Indeed, $f(x)$ is guaranteed to be within $\varepsilon$ of 0 provided $x$ is within __________ of 0.

(b) [1 mark] Again, consider the function $f(x)$ whose graph is pictured above. Fill in the blank so that the following is a correct proof that $\lim_{x \to 8} f(x)$ does not exist.

Suppose $\lim_{x \to 8} f(x) = L$. Let $\varepsilon$ be equal to __________. Can we guarantee $f(x)$ to be within $\varepsilon$ of $L$ provided $x$ is sufficiently close to 8? We cannot. For no matter how small we make the interval $(8 - \delta, 8 + \delta)$, it includes $x$-values such that $f(x)$ is not within $\varepsilon$ of $L$.
(c) [2 marks] Evaluate the limit \( \lim_{x \to \infty} \frac{\sqrt{100x^4 + x}}{2x^2 - \sqrt{\pi}}. \)

(d) [2 marks] Determine if the series \( \sum_{n \geq 3} \frac{n^{3/2}}{n^2 + n + 5} \) converges.
(e) [2 marks] Determine if the series $\sum_{n \geq 1} \frac{(-1)^n n^3}{3^n}$ converges.

(f) [2 marks] Define what it means for a function $f(x)$ to be continuous at $x = a$.

(g) [2 marks] Write down an algebraic expression for a function which is not continuous at $x = 5$. Then sketch the graph of the function.
2. For each of the following statements, determine if it is true. If it is true, provide a brief justification. If it is false, provide a counterexample.

(a) [2 marks] If $f(x)$ is continuous, then $f(x)$ is differentiable.

(b) [2 marks] If $\{a_n\}$ converges to $0$, then $\{(-1)^n a_n\}$ converges to $0$.

(c) [2 marks] If $\lim_{n \to \infty} a_n = 0$, then $\sum_{n \geq 1} a_n$ converges.
3. (a) [2 marks] State the Comparison Test.

(b) [3 marks] Let \( \{a_n\} \) be a sequence with \( 0 \leq a_n \leq b \) for all \( n \), where \( b < 2 \) is a constant. Prove that \( \sum_{n \geq 1} \frac{a_n^{2n+1}}{4^n} \) converges.

(c) [1 mark] Let \( \{a_n\} \) be a sequence with \( 0 \leq a_n \leq c \) for all \( n \), where \( c \leq 2 \) is a constant. Explain why \( \sum_{n \geq 1} \frac{a_n^{2n+1}}{4^n} \) need not converge.
4. Imagine you are standing in the middle of a large field. You walk 16 metres in one direction, turn 90° to the right and walk 8 metres, turn 90° to the right and walk 4 metres, 90° to the right and walk 2 metres, and so on. This continues indefinitely. The few turns are pictured below, with horizontal and vertical paths.

(a) [1 mark] Write down a series describing your total horizontal displacement.

(b) [1 mark] Write down a series describing your total vertical displacement.

(c) [1 mark] Calculate your total horizontal displacement by determining what the series in part (a) converges to.

(d) [1 mark] Calculate your total vertical displacement by determining what the series in part (b) converges to.

(e) [1 mark] As the number of turns you take approaches infinity, you will get arbitrarily close to a point $P$. Find the distance between your original position and $P$. 
5. Let \( P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \) be a polynomial where \( n \) is odd and \( a_{n-1}, \ldots, a_1, a_0 \) are constants. In this question you will prove that \( P(x) \) has a root.

(a) [2 marks] State the Intermediate Value Theorem.

(b) [2 marks] Let \( Q(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \). Explain why \( \lim_{x \to \pm\infty} \left( 1 + \frac{Q(x)}{x^n} \right) = 1 \).

(c) [2 marks] Explain why there must exist numbers \( a < 0 \) and \( b > 0 \) such that \( P(a) < 0 \) and \( P(b) > 0 \). (Hint: Note that \( P(x) = x^n \left( 1 + \frac{Q(x)}{x^n} \right) \) for all \( x \neq 0 \).)

(d) [1 mark] Explain why we may conclude that \( P(x) \) has a root.
6. [4 marks] Let $f(x)$ be a nonzero differentiable function. Prove using the definition of derivative that
\[ \frac{d}{dx} \left( \frac{1}{f(x)} \right) = -\frac{f'(x)}{f(x)^2}. \]
(You must use the definition of derivative; no credit will be given for proving the result using the Quotient Rule or Chain Rule.)