Rules governing formal examinations:

Each candidate must be prepared to produce, upon request, a UBC card for identification.

No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:

- Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
- Speaking or communicating with other candidates;
- Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.

Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

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Note that your answers must be in “calculator-ready” form, but they do not have to be simplified.

In general, you may use any result proven in class or on assignments. You do not have to use the precise definition of limit or the definition of derivative, unless it is specifically requested, as in questions 1(a) and 1(d).

This page may be used for rough work. It will not be marked.
1. (a) [2 marks] Let \( \{a_n\} \) and \( \{b_n\} \) be sequences such that \( 0 \leq a_n \leq b_n \) for all \( n \). Suppose \( \{b_n\} \) converges to 0. Prove using the definition of limit that \( \{a_n\} \) converges to 0.

(b) [2 marks] Determine all the values of \( a \) for which \( \sum_{n \geq 1} \frac{a^{n-1}}{2^n} \) converges.

(c) [2 marks] State the Ratio Test.
(d) [2 marks] State the limit definition for what it means for \( f(x) \) to be differentiable at a point \( a \).

(e) [2 marks] Find the equation of the line tangent to \( y = \sqrt{x} \) at \((4,2)\).

(f) [2 marks] Recall that a curve is said to be nonsingular if a single tangent line may be drawn at any point on the curve. Explain in one or two sentences why the curve \( y^2 = x \) is nonsingular.
(g) [3 marks] Prove that the curve \( y + e^x = \cos(\sqrt{x}) + 2 \) crosses the \( x \)-axis at least once.

(h) [2 marks] Write down an algebraic expression for a function which has at least one horizontal asymptote and at least one vertical asymptote. Then sketch the graph of the function.

(i) [2 marks] Write down an algebraic expression for a function which is concave down and increasing everywhere. Then sketch the graph of the function.
2. For each of the following statements, determine if it is true. If it is true, provide a brief justification. If it is false, provide a counterexample.

(a) [2 marks] If \(\{a_n\} \) converges and \(\{a_n\} \) converges, then \(\{b_n\} \) converges.

(b) [2 marks] If \(|a_n| < A\) for a constant \(A\) and all \(n\), then \(\sum_{n \geq 1} \frac{(-1)^n a_n}{2^n} \) converges.

(c) [2 marks] If \(f(x)\) is continuous everywhere and has a global minimum at \(x = 2\), then \(f'(2) = 0\).
(d) [2 marks] If $f(x)$ is differentiable everywhere, then $f(x)$ has no vertical asymptotes.

(e) [2 marks] If $f(x)$ is infinitely differentiable, then its $n^{th}$ derivative $f^{(n)}(x)$ is equal to 0 for some $n$.

(f) [2 marks] If $f(0) = -1$ and $f(1) = 2$, then $f'(a) = 3$ for some number $a$ in $(0, 1)$. 
3. Determine if the following series converge.

(a) [2 marks] \[ \sum_{n \geq 1} \frac{(-1)^n n^3}{3^n}. \]

(b) [2 marks] \[ \sum_{n \geq 1} \cos \left( e^{-n!} \right). \]

(c) [2 marks] \[ \sum_{n \geq 1} \frac{1 + 3^n}{4^n}. \]
4. Differentiate the following functions.

(a) [2 marks] \( f(x) = \frac{x^2}{x + 1} \).

(b) [2 marks] \( f(x) = x^2 \cos \left( \frac{1}{x^2} \right) \).

(c) [2 marks] \( f(x) = x^x \).
5. Let \( f(x) = \frac{1}{x} \).

(a) \([2 \text{ marks}]\) Calculate the first four derivatives of \( f(x) \): \( f'(x) \), \( f''(x) \), \( f^{(3)}(x) \), and \( f^{(4)}(x) \).

(b) \([1 \text{ mark}]\) Fill in the following conjecture: \( f^{(n)}(x) = \boxed{} \).

(c) \([1 \text{ mark}]\) Explain in one or two sentences why your conjecture is true for \( n = 1 \).

(d) \([2 \text{ marks}]\) Prove that if your conjecture is true for \( n = k \), then it is true for \( n = k + 1 \).

(e) \([1 \text{ mark}]\) Explain in one or two sentences why you may conclude that your conjecture is true for all positive integers \( n \).
6. The curve \( x^{2/3} + y^{2/3} = 1 \) is an example of an \textit{astroid}. It is pictured below, along with the line \( L \) tangent to the curve at \((x_0, y_0)\) in the first quadrant.

(a) [2 marks] Find the equation of \( L \).

(b) [2 marks] Prove that \( L \) has \( x \)-intercept \( x_0^{1/3} \).

(c) [2 marks] Prove that \( L \) has \( y \)-intercept \( y_0^{1/3} \).

(d) [2 marks] Prove that the segment of \( L \) contained in the first quadrant is of length 1.
7. [7 marks] Consider two dots, $D_1$ and $D_2$, on wheels of radius 2 metres. The wheels are set a distance of 1 metre apart. The dots start opposite each other, at a distance of 1 metre, when the wheels begin to turn in opposite directions at a constant rate of 1 full rotation every 5 seconds. The figure below pictures the dots shortly after the wheels begin to turn.

Let $H$ be the horizontal distance between the two dots. Find the rate at which $H$ is increasing 1 second after the wheels begin to turn. (Hint: describe $H$ using $\theta$, the angle through which each of the wheels has turned.)
8. Let \( f(x) = (x + 2)(x + 1)e^{-x} \).

(a) [1 mark] State the domain of \( f(x) \).

(b) [2 marks] Find the \( x \)-intercept(s) and \( y \)-intercept of \( f(x) \), if any exist.

(c) [2 marks] Determine if \( f(x) \) has any horizontal asymptotes; and if so, what they are.

(d) [1 mark] Determine if \( f(x) \) has any vertical asymptotes; and if so, what they are.
(e) [5 marks] Determine where $f(x)$ is increasing, and where it is decreasing.

(f) [5 marks] Determine where $f(x)$ is concave up, and where it is concave down.
(g) [3 marks] Sketch a large graph of \( f(x) \), using the information determined in parts (a) through (f). Make sure to label the \( x \)-values of all critical points and inflection points. (You do not have to label their \( y \)-values.)
9. [6 marks] Find the area of the largest isosceles triangle that can be inscribed inside a circle of radius $r$. (Recall that this circle is described by the equation $x^2 + y^2 = r^2$.)