1. The Mean Value Theorem implies that if \( f'(x) > 0 \) on an interval, then \( f(x) \) is increasing on that interval. It is crucial that the condition hold on an interval, not just at a single point. In this question, you will demonstrate this by analyzing a function that has a positive derivative at a single point but is not increasing on any interval containing that point. Let
\[
f(x) = \begin{cases} 
x + 4x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
\]

(a) Prove that \( f'(0) = 1 \).
(b) Prove that in any interval containing \( x = 0 \) there is an interval in which \( f(x) \) is decreasing.

(a) We have
\[
f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \left( 1 + 4h \sin \left( \frac{1}{h} \right) \right) = 1.
\]
(The last inequality follows from the observation that
\[
1 - 4h \leq 1 + 4h \sin \left( \frac{1}{h} \right) \leq 1 + 4h,
\]
with the upper and lower bounds tending toward 1 as \( h \to 0 \).)

(b) When \( x \neq 0 \), we have \( f'(x) = 1 + 8x \sin \left( \frac{1}{x} \right) - 4 \cos \left( \frac{1}{x} \right) \). Note that on the interval \(-\frac{1}{8} \leq x \leq \frac{1}{8}\) we have
\[
-1 \leq 8x \sin \left( \frac{1}{x} \right) \leq 1,
\]
whence
\[
-4 \cos \left( \frac{1}{x} \right) \leq f'(x) \leq 2 - 4 \cos \left( \frac{1}{x} \right).
\]
We can “force” \( f'(x) < 0 \) by restricting our \( x \)-values to intervals around \( \frac{1}{2(k+1)\pi} \) that are small enough to make \( 4 \cos \left( \frac{1}{x} \right) \) much larger than 2. For example, for all \( x \)-values in intervals of the form \( \left( \frac{1}{(2k+1)\pi + \frac{\pi}{2}}, \frac{1}{(2k+1)\pi + \frac{3\pi}{2}} \right) \), we have
\[
4 \cos \left( \frac{1}{x} \right) \geq 4 \left( \frac{\sqrt{3}}{2} \right) = 2\sqrt{3}.
\]
It remains only to observe that there are intervals of the form \( \left( \frac{1}{(2k+1)\pi + \frac{\pi}{2}}, \frac{1}{(2k+1)\pi + \frac{3\pi}{2}} \right) \) arbitrarily close to \( x = 0 \).

2. (a) Suppose \( f(x) \) is twice-differentiable with \( n \) roots, where \( n \geq 3 \). Prove that \( f''(x) = 0 \) at least \( n - 2 \) times.
(b) Is the converse of the statement in part (a) true? In other words, suppose \( f(x) \) is twice-differentiable with \( f''(x) = 0 \) \( m \) times, where \( m \geq 1 \). Is it true that \( f(x) \) has \( m + 2 \) roots? Prove that it is, or provide a counterexample.
(a) Suppose $f(x)$ has roots $r_1 < r_2 < \cdots < r_n$. By Rolle’s Theorem, there exist points $s_1, s_2, \ldots, s_{n-1}$ in the intervals $(r_1, r_2), (r_2, r_3), \cdots, (r_{n-1}, r_n)$, respectively, with $f'(s_1) = f'(s_2) = \cdots = f(s_{n-1}) = 0$. Applying Rolle’s Theorem to the function $f'(x)$, there exist points $t_1, t_2, \ldots, t_{n-2}$ in the intervals $(s_1, s_2), (s_2, s_3), \cdots, (s_{n-2}, s_{n-1})$, respectively, with $f''(t_1) = f''(t_2) = \cdots = f''(t_{n-2}) = 0$.

(b) The converse is not true. Take, for instance the function $f(x) = x^3 + 3$. $f''(x) = 0$ exactly once, at $x = 0$; but $f(x)$ has one root, not three.