1. In recitations, you showed a limited version of the Power Rule; namely, that \( \frac{d}{dx} x^n = x^{n-1} \) for nonnegative integers \( n \). In this question, you extend the Power Rule to functions of the form \( x^{n/2} \) where \( n \) is a positive integer. You may not use any other version of the Power Rule in your proof.

(a) Prove that \( \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{1/2 - 1} \).

(b) Prove that if \( \frac{d}{dx} x^{k/2} = \frac{k}{2} x^{k/2 - 1} \) for a positive integer \( k \), then \( \frac{d}{dx} x^{(k+1)/2} = \frac{k+1}{2} x^{(k+1)/2 - 1} \).

(c) Explain in a few sentences why your work in parts (a) and (b) imply that \( \frac{d}{dx} x^{n/2} = \frac{n}{2} x^{n/2 - 1} \) for all positive integers \( n \).

(a) We use the limit definition of derivative for part (a):

\[
\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h \left( \sqrt{x+h} + \sqrt{x} \right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
\]

(b) Now assume \( \frac{d}{dx} x^{k/2} = \frac{k}{2} x^{k/2 - 1} \) for a positive integer \( k \). Then

\[
\frac{d}{dx} x^{(k+1)/2} = \frac{d}{dx} (\sqrt{x} x^{k/2}) = \left( \frac{1}{2\sqrt{x}} x^{k/2} + \sqrt{x} \frac{d}{dx} x^{k/2} \right) = \frac{1}{2\sqrt{x}} x^{k/2} + \sqrt{x} \frac{k}{2} x^{k/2 - 1} = \frac{1}{2} \left( x^{(k-1)/2} + k x^{(k-1)/2} \right) = \frac{k+1}{2} x^{(k+1)/2 - 1},
\]

as needed. (The second equality follows from the Product Rule and part (a), while the third equality follows from the induction hypothesis.)

(c) In part (a), we proved the result stated in part (c) for \( n = 1 \) directly. By part (b), this implies that the result stated in part (c) holds for \( n = 2 \). Again by part (b), this implies that the result holds for \( n = 3 \), which implies that it holds for \( n = 4 \), and so on. This chain of implications, started by part (a) and carried on by part (b), extends to all positive integers \( n \).

2. Choose the question on the midterm that was the hardest for you, and write a “model answer” for it. This should be an answer that is not only correct, but provides full explanations and a summary of the “key observation” you needed to make to solve the problem correctly.