ASSIGNMENT 5

Solutions

Let \( f(x) \) be continuous on \([l, r]\), \( f(l) < f(r) \), and \( L \in (f(l), f(r)) \) be a number between \( f(l) \) and \( f(r) \). In this assignment you will provide two different proofs that there exists a number \( a \in (l, r) \) such that \( f(a) = L \).

1. Consider the following algorithm.
   (a) Divide \([l, r]\) in half, and call the midpoint \( m \). If \( f(m) = L \), we are done. Otherwise, if \( f(m) > L \), let \([l, m]\) be the new interval, and if \( f(m) < L \), let \([m, r]\) be the new interval. Name the new interval \([l_1, r_1]\).
   (b) Divide \([l_1, r_1]\) in half, and call the midpoint \( m_1 \). If \( f(m_1) = L \), we are done. Otherwise, if \( f(m_1) > L \), let \([l_1, m_1]\) be the new interval, and if \( f(m_1) < L \), let \([m_1, r_1]\) be the new interval. Name the new interval \([l_2, r_2]\).
   (c) Repeat the process \(ad infinitum\), getting a sequence of nested intervals
   \[ [l_1, r_1] \supset [l_2, r_2] \supset [l_3, r_3] \supset \cdots \]
   Explain carefully why \( \lim_{n \to \infty} f(l_n) = \lim_{n \to \infty} f(r_n) = L \).

First, the sequence \( \{l_n\} \) is increasing and bounded above (by \( r_1 \), say) which means it converges. Suppose it converges to a number \( c \in [l, r] \). By the continuity of \( f(x) \) on that interval, we have
\[ \lim_{x \to c} f(x) = \lim_{n \to \infty} f(l_n) = f(c) \leq L, \]
with the last inequality following from the fact that \( f(l_n) < L \) for all \( n \). In a similar way, we may conclude that \( \{r_n\} \) converges to a number \( d \in [l, r] \), and
\[ \lim_{x \to d} f(x) = \lim_{n \to \infty} f(r_n) = f(d) \geq L. \]
It remains to show only that \( c = d \). This follows from the fact that we are halving our intervals' length each iteration of the algorithm, whence
\[ \lim_{n \to \infty} (r_n - l_n) = \lim_{n \to \infty} \frac{r - l}{2^n} = 0. \]
On the other hand, we also have
\[ \lim_{n \to \infty} (r_n - l_n) = \lim_{n \to \infty} r_n - \lim_{n \to \infty} l_n = d - c. \]

2. Let \( S \) be the set of all \( x \in [l, r] \) such that \( f(x) \leq L \), and let \( a \) be the least upper bound of \( S \).
   (a) Prove by contradiction that \( f(a) \leq L \).
   (b) Prove by contradiction that \( f(a) \geq L \).

   (a) Suppose \( f(a) = M > L \). By the continuity of \( f(x) \), \( \lim_{x \to a} f(x) = M \), which means that there exists some \( \delta > 0 \) such that if \( x \in (a - \delta, a + \delta) \), then \( f(x) > L \). Taking just one of these \( x \)-values that is smaller than \( a \) — say \( a - \frac{\delta}{2} \) — contradicts the fact that \( a \) is the least upper bound on \( x \)-values such that \( f(x) \leq L \), since \( f \left( a - \frac{\delta}{2} \right) > L \).

   (b) Now suppose \( f(a) = K < L \). \( \lim_{x \to a} f(x) = K \), which means that there exists some \( \delta > 0 \) such that if \( x \in (a - \delta, a + \delta) \), then \( f(x) < L \). Taking just one of these \( x \)-values that is larger than \( a \) — say \( a + \frac{\delta}{2} \) — contradicts the fact that \( a \) is an upper bound on \( x \)-values such that \( f(x) \leq L \), since \( f \left( a + \frac{\delta}{2} \right) < L \).