1. Explain how the terms in $\sum_{n \geq 1} \frac{(-1)^{n-1}}{n}$ may be rearranged so that the series converges to $\pi$.

Consider the subsequences $\{a_n\} = \{1, \frac{1}{3}, \frac{1}{5}, \ldots\}$ and $\{b_n\} = \{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \ldots\}$ of the sequence of terms in the alternating harmonic series. We rearrange the series as follows.

(a) Add enough terms from the beginning of $\{a_n\}$ so that the sum is “just larger” than $\pi$ (so if we added one less term, our sum would be smaller than $\pi$). Then remove those terms from $\{a_n\}$.

(b) Add enough terms from the beginning of $\{b_n\}$ so that the sum is “just smaller” than $\pi$. Then remove those terms from $\{b_n\}$.

(c) Return to step (a).

The series then consists of strings of positive terms, say of lengths $r_1, r_2, r_3, \ldots$, alternating with strings of negative terms, say of lengths $s_1, s_2, s_3, \ldots$, with each string bringing the partial sums “just past” $\pi$. We claim that this rearrangement converges to $\pi$.

This is because the first $r_1$ terms bring the partial sum to within $a_{r_1}$ of $\pi$. The next $s_1$ terms bring the partial sum to within $-b_{s_1}$ of $\pi$. The next $r_2$ terms bring the partial sum to within $a_{r_2}$ of $\pi$. The next $s_2$ terms bring the partial sum to within $-b_{s_2}$ of $\pi$, and so on.

Since the subsequences $\{a_n\}$ and $\{b_n\}$ both converge to 0, the “sub-subsequences” $\{a_{r_n}\}$ and $\{b_{s_n}\}$ also converge to 0. In other words, the differences between the partial sums of the rearrangement and $\pi$ also converge to 0.

2. Your work in the previous question implies that all conditionally convergent series may be rearranged to converge to any limit (or indeed to diverge). It turns out that rearranging any absolutely convergent series does not affect its convergence behaviour. In this question, we prove a limited version, that rearranging any positive convergent series does not affect its convergence behaviour.

Let $\sum_{n \geq 1} a_n$ be a convergent series with all positive terms, and let $\sum_{n \geq 1} b_n$ be a rearrangement of that series.

(a) Prove that $\sum_{n \geq 1} b_n$ converges.

(b) Suppose $\sum_{n \geq 1} a_n = A$ and $\sum_{n \geq 1} b_n = B$. Prove that $A = B$.

(a) Let $\{B_n\}$ be the sequence of partial sums of $\sum_{n \geq 1} b_n$. $\{B_n\}$ is increasing, since every term $b_n$ is positive.

It is also bounded, since every term in $B_n$ is in the sum $\sum_{n \geq 1} a_n$, whence $B_n \leq A$. Thus $\{B_n\}$ converges by the Bounded Monotone Convergence Theorem.
(b) Taking the limit of the inequality \( B_n \leq A \) yields
\[
\lim_{n \to \infty} B_n = B \leq A.
\]

However, we can do the same thing for the sequence \( \{A_n\} \) of \( n^{th} \) partial sums of \( \sum_{n \geq 1} a_n \). Since every term in \( A_n \) is in the sum \( \sum_{n \geq 1} b_n \), we have \( A_n \leq B \) for all \( n \). Taking the limit, we have
\[
\lim_{n \to \infty} A_n = A \leq B.
\]

Therefore \( A = B \).