1. Let \( \{a_n\} \) be the ordered sequence of positive integers not containing the digit “0”; that is, \( \{a_n\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, \ldots\} \).

Prove that \( \sum_{n \geq 1} a_n \) converges.

The sequence of partial sums of the series is clearly increasing. We claim that it is bounded as well. Let \( S_i \) denote the set of \( i \)-digit numbers not containing the digit “0”. \( S_i \) has \( 9^i \) elements, since each digit may be selected from the set \( \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \). Moreover, any number \( s \) in \( S_i \) satisfies \( x \geq 10^i - 1 \). Thus

\[
\sum_{n \geq 1} a_n = \sum_{i \geq 1} \sum_{s \in S_i} \frac{1}{s} \leq \sum_{i \geq 1} \frac{9^i - 1}{10^i - 1} = 90.
\]

2. When applying the Comparison and Limit Comparison Tests, it is often useful to compare series to \( p \)-series of the form \( \sum_{n \geq 1} \frac{1}{n^p} \) where \( p > 0 \). However, we have not examined the convergence of any \( p \)-series other than the harmonic series.

In this question, we determine the convergence of \( p \)-series for \( p \geq 2 \). (It remains to determine the convergence for \( 1 < p < 2 \), which we will do after we define the integral next term.)

Let \( \sum_{n \geq 1} a_n \) be a series with all positive terms satisfying

\[
\lim_{n \to \infty} n \left(1 - \frac{a_{n+1}}{a_n}\right) = L > 1. \quad (1)
\]

(a) Let \( r \) be such that \( 1 < r < L \). Prove that there exists some positive integer \( M \) such that

\[(r - 1)a_n < (n - 1)a_n - na_{n+1} \quad \text{for} \quad n \geq M.\]

(Hint: interpret (1) with respect to \( r \).)

(b) Let \( N > M \) be an integer. Using part (a), prove that

\[(r - 1) (a_M + a_{M+1} + \cdots + a_N) < (M - 1)a_M.\]

(c) Using part (b) and the Bounded Monotone Convergence Theorem, prove that \( \sum_{n \geq 1} a_n \) converges.

(d) Using the results shown in the previous three parts, prove that \( \sum_{n \geq 1} \frac{1}{n^p} \) converges for \( p \geq 2 \).
(a) Since \( \lim_{n \to \infty} n \left( 1 - \frac{a_{n+1}}{a_n} \right) = L \), and since \( r < L \), the terms on the left-hand side of the equation are eventually within \( L - r \) of \( L \); that is, there exists a positive integer \( M \) such that, for \( n \geq M \),

\[
n \left( 1 - \frac{a_{n+1}}{a_n} \right) > r,
\]

which after expanding, multiplying by \( a_n \) and subtracting \( a_n \) from both sides, is equivalent to

\[
(r - 1)a_n < (n - 1)a_n - na_{n+1}.
\]

(b) By part (a), we have

\[
(r - 1)a_M < (M - 1)a_M - Ma_{M+1},
\]

\[
(r - 1)a_{M+1} < Ma_{M+1} - (M + 1)a_{M+2},
\]

\[
(r - 1)a_{M+2} < (M + 1)a_{M+2} - (M + 2)a_{M+3},
\]

\[
\vdots
\]

\[
(r - 1)a_{N-1} < (N - 2)a_{N-1} - (N - 1)a_N,
\]

\[
(r - 1)a_N < (N - 1)a_N - Na_{N+1}.
\]

Taking the sum, which is telescoping on the right-hand side, we get

\[
(r - 1) (a_M + a_{M+1} + \cdots + a_N) < (M - 1)a_M - Na_{N+1} < (M - 1)a_M.
\]

(c) The partial sums of \( \sum_{n \geq 1} a_n \) are clearly increasing. By part (b), the \( N \)th partial sum has the bound

\[
\sum_{n=1}^{N} a_n = a_1 + \cdots + a_{k-1} + \sum_{n=M}^{N} a_n < a_1 + \cdots + a_{k-1} + \left( \frac{M - 1}{r - 1} \right) a_M.
\]

Note that the right-hand side does not depend on \( N \). Thus the partial sums are increasing and bounded, and the series converges.

(d) We show that \( \sum_{n \geq 1} \frac{1}{n^2} \) converges by observing that

\[
\lim_{n \to \infty} n \left( 1 - \frac{1}{\left( \frac{(n+1)^2}{n^2} \right)} \right) = 2 > 1.
\]

Hence \( \sum_{n \geq 1} \frac{1}{n^p} \) converges for \( p \geq 2 \), by the Comparison Test.