ASSIGNMENT 10

Solutions

1. In this question you will sketch the curve $C$ described by the equation $x^2 - y^2 = 1$.
   
   (a) Describe and justify the domain of $C$.
   
   (b) Write down the $x$- and $y$-intercepts of $C$.
   
   (c) Use implicit differentiation to determine where $C$ is increasing and where it is decreasing. Note that you will have to describe this using more than just $x$-values, since an $x$-value on the domain may yield two $y$-values.
   
   (d) Use implicit differentiation to determine where $C$ is concave up and where it is concave down.
   
   (e) Draw a large sketch of $C$.

(a) Note that $x^2 = 1 + y^2 \geq 1,$

which implies that $C$ has domain $(-\infty, -1] \cup [1, \infty)$.

(b) We have $x$-intercepts $(-1, 0)$ and $(0, -1)$. There are no $y$-intercepts.

(c) Differentiating $x^2 - y^2 = 1$ yields $2x - 2y \frac{dy}{dx} = 0$; that is, $\frac{dy}{dx} = \frac{x}{y}$. This means that $C$ is increasing in the first quadrant, where $x, y > 0$; and in the third quadrant, where $x, y < 0$. $C$ is decreasing in the second and third quadrants where $x$ and $y$ have different signs.

(d) Taking the second derivative of $\frac{dy}{dx} = \frac{x}{y}$ yields

$$\frac{d^2 y}{dx^2} = \frac{y - x \frac{dy}{dx}}{y^2} = \frac{y - \frac{x^2}{y}}{y^2} = \frac{y^2 - x^2}{y^3} = -\frac{1}{y^3}.$$  

This implies that $C$ is concave down on the half-plane $y > 0$, and concave up on the half-plane $y < 0$.

(e) $C$ is a hyperbola. We may add to our information in parts (a) and (d) the fact that $C$ has slant asymptotes $y = x$ and $y = -x$. The graph is below.
2. Suppose you have four points on the corners of a square of side length $L$. Construct a path of minimal length that connects all four points. You must justify your path arrangement, but you may assume without proof that the shortest distance between two points on a plane is a straight line.

It is beyond the scope of this course to prove a minimal configuration, but we gesture at one here. Imagine the square is centred on the origin with each of the sides parallel to an axis. A configuration that is not symmetrical about either axis is non-optimal — we can simply replace the less optimal side with a copy of the more optimal side.

We end up considering two configurations, illustrated below.

These may be seen as the endpoints of the configuration below, which we now optimize with respect to $x$.

We wish to minimize

$$P(x) = (L - 2x) + 4\sqrt{x^2 + \frac{L^2}{4}} = (L - 2x) + 2\sqrt{4x^2 + L^2}$$

on the domain $[0, \frac{L}{2}]$. Differentiating, we get $P'(x) = -2 + \frac{8x}{\sqrt{4x^2 + L^2}}$, which vanishes when $x = \frac{L}{\sqrt{12}}$. (We discard the negative solution since it is not in the domain.) By direct calculation, we determine this to be a global minimum on the domain.

(It is a significant fact that, at the minimum, paths meet inside the square at angles of $\frac{2\pi}{3}$.
For more on this, look up “Steiner tree problem”.)