Due to how difficult the problem can appear to be because of some subtle points involved, I have written up an in-depth solution for the substantial part of Assignment 1 Problem 1. Note that only the Claim and the Proof are actually necessary to be considered a good solution.

Don’t let the length intimidate you - The proof itself is only one page (pg. 4).

There also exist many solutions, and this is just one example. Some comments on this have been included near the back, after the proof.

Claim:

\[ \lim_{x \rightarrow 2^+} \frac{1}{x^2 - 4} = \infty. \]

Aside 1 (Definitions):

Note the definition of a vertical asymptote: A function \( f(x) \) has a vertical asymptote at \( x = a \) if \( \lim_{x \rightarrow a^-} f(x) = \infty \) or \( \lim_{x \rightarrow a^+} f(x) = \infty \). (Both could be true, but we only need one to be true to say that there is a vertical asymptote at \( x = a \).)

This needs us to also understand the definitions of these limits. We say

\[ \lim_{x \rightarrow a^+} f(x) = \infty \]

when we can make \( f(x) \) arbitrarily large by making \( x \) sufficiently close to \( a \) on the right. To be more exact, for any \( N > 0 \) there exists some \( \delta > 0 \) such that if we have \( 0 < x - a < \delta \), then we certainly get \( f(x) > N \). (The definition for \( x \rightarrow a^- \) is the same, except with \( 0 < a - x < \delta \).)
It is always a good idea to write out our claim in the form of the definition. In our case, the claim can be read as:

For any $N > 0$ there exists some $\delta > 0$ such that

if $0 < x - 2 < \delta$, then $\frac{1}{x^2 - 4} > N$.

Aside 2 (Rough Work):

Our proof will begin with a clever choice of $\delta$ and end with the conclusion that if $0 < x - 2 < \delta$, then $\frac{1}{x^2 - 4} > N$.

The rough work is the part where we come up with the $\delta$. Often we will find that we begin with the conclusion (what we want) and work backwards to find $\delta$.

In the proof, we will let some $N > 0$ be given. So we want

$$\frac{1}{x^2 - 4} > N$$

and we will have some control over the size of $x - 2$, so a useful form is

$$\frac{1}{(x - 2)(x + 2)} > N$$

or since we will assume $x - 2 > 0$ in the proof later,

$$(x - 2)(x + 2) < \frac{1}{N}.$$  

(Subtle) Now notice that we can make $\delta$ as small as we like, as long as it is already small enough. This means that I can demand

$$\delta \leq 1.$$  

After all, if we find that something like $\delta = 10$ is sufficiently small, there is nothing
ever wrong with shrinking \( \delta \) further. Smaller than 'small enough' will still work! On the other hand, if \( \delta \) needs to be much tinier than 1, then \( \delta \leq 1 \) is already true, so we do not need to do anything.

During the proof, \( x \) will satisfy \( 0 < x - 2 < \delta \leq 1 \) by the above demand. What about \( x + 2 \), though? Well, we can just add 4 to \( x - 2 < 1 \) to get \( x + 2 < 5 \).

Now we have

\[
(x - 2)(x + 2) < (1)(5) = 5.
\]

Then if \( \frac{1}{N} \geq 5 \) (or \( N \leq \frac{1}{5} \)), we would successfully have shown that

\[
(x - 2)(x + 2) < 5 \leq \frac{1}{N}.
\]

What if \( N > \frac{1}{5} \)? We still have \( x - 2 < \delta \) and \( x + 2 < 5 \), so let us go back a bit:

\[
(x - 2)(x + 2) < \delta \cdot 5.
\]

We would like the right end to be less than or equal to \( \frac{1}{N} \). This is easy to do.

Once again, **demand** that

\[
\delta \leq \frac{1}{5N}.
\]

Remember, this is perfectly fine to do, because the right side is positive.

Now we have two conditions for \( \delta \). These are \( \delta \leq 1 \) and \( \delta \leq \frac{1}{5N} \). An easy \( \delta \) is

\[
\delta = \min \left(1, \frac{1}{5N}\right).
\]

An equivalent way to write this would be
\[
\delta = \begin{cases} 
1, & \text{if } N \leq \frac{1}{5} \\
\frac{1}{5N}, & \text{if } N > \frac{1}{5}
\end{cases}
\]

We are now prepared to write the actual proof.

**Proof:**

Let some \( N > 0 \) be given. Choose \( \delta = \min \left(1, \frac{1}{5N}\right) \).

Let \( x \) be such that \( 0 < x - 2 < \delta \). Then \( x + 2 < \delta + 4 \). Multiplying these together gives \((x - 2)(x + 2) = x^2 - 4 < \delta(\delta + 4)\). It follows that

\[
\frac{1}{x^2 - 4} > \frac{1}{\delta(\delta + 4)}.
\]

Clearly, either \( N \leq \frac{1}{5} \) or \( N > \frac{1}{5} \). Call these cases 1 and 2, respectively.

**Case 1:** Suppose \( N \leq \frac{1}{5} \). Then \( \frac{1}{5N} \geq 1 \) so \( \delta = 1 \). It follows that

\[
\frac{1}{x^2 - 4} > \frac{1}{\delta(\delta + 4)} = \frac{1}{(1)(1 + 4)} = \frac{1}{5} \geq N.
\]

**Case 2:** Suppose \( N > \frac{1}{5} \). Then \( \frac{1}{5N} < 1 \) so \( \delta = \frac{1}{5N} \). It follows that

\[
\frac{1}{x^2 - 4} > \frac{1}{\delta(\delta + 4)} = \frac{1}{\frac{1}{5N}((\frac{1}{5N}) + 4)} = \frac{5N}{\frac{1}{5N} + 4},
\]

but \( \frac{1}{5N} < 1 \) so

\[
\frac{1}{x^2 - 4} > \frac{5N}{\frac{1}{5N} + 4} > \frac{5N}{(1) + 4} = N.
\]

In either case, if \( 0 < x - 2 < \delta \) we satisfy \( \frac{1}{x^2 - 4} > N \). In conclusion,

\[
\lim_{x \to 2^+} \frac{1}{x^2 - 4} = \infty.
\]
Aside 3 (Comments):

In the 'subtle' move of demanding $\delta \leq 1$, the 1 isn’t all that special (It does make the algebra easier, though.). Try the same proof by demanding $\delta \leq b$, where $b$ is a number you like! How does your choice of $\delta$ change?

Notice that whenever feel like you have several $\delta$’s you want to choose from, you can just define the actual $\delta$ to be the minimum of all those choices. If $\delta = \min(a, b, c)$, then $\delta \leq a$, $\delta \leq b$, and $\delta \leq c$ are all true.

For this problem, there also is a choice of $\delta$ that doesn’t require the use of min. It’s fine to solve this problem that way, but it is recommended that you know how to use min for other problems.