

Second Order Linear Differential Equations

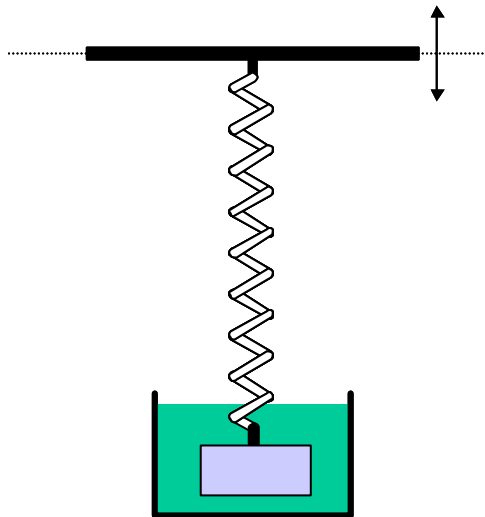
Equations like:

$$\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = g(t) \quad (1)$$

Engineering examples:

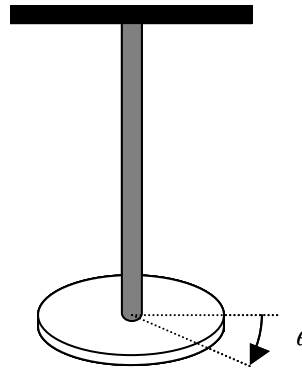
1. Spring-dashpot systems:

$$m \frac{d^2 x}{dt^2} = -kx - \beta \frac{dx}{dt} + f(t)$$



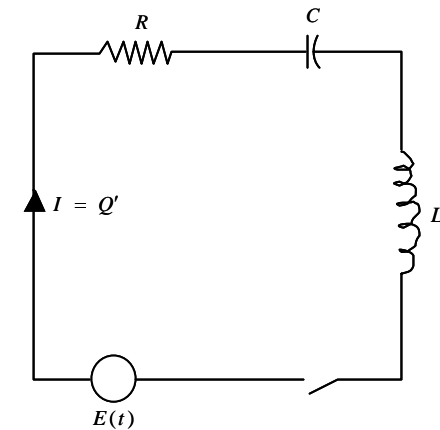
2. Torsional motion of a weight on a twisted shaft:

$$I \frac{d^2 \theta}{dt^2} + c \frac{d\theta}{dt} + k\theta = T(t)$$



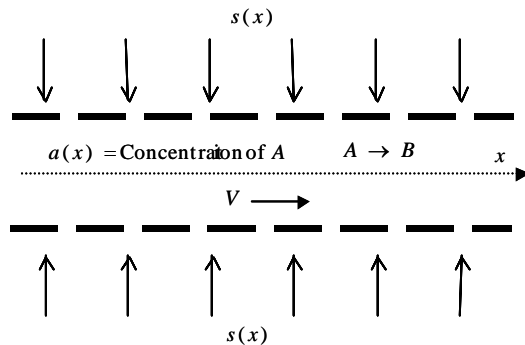
3. L-R-C series circuits:

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = E(t)$$



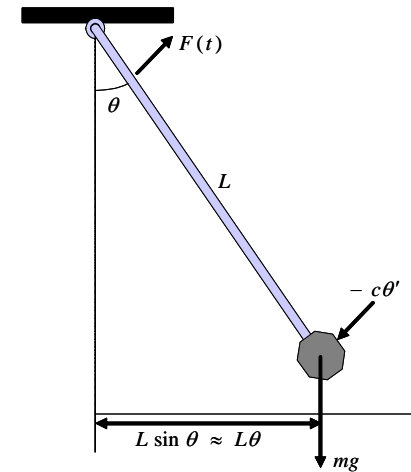
4. Diffusion-convection-reaction of a chemical in a permeable channel: $A \rightarrow B$

$$D \frac{d^2 a}{dx^2} - V \frac{da}{dx} - ka - s(x) = 0$$



5. Small oscillations of a pendulum:

$$mL^2 \frac{d^2 \theta}{dt^2} = -cL \frac{d\theta}{dt} - mgL\theta + F(t)$$



These 5 physical systems all have identical mathematical form:

$$Ay'' + By' + Cy = D(t)$$

System	A	B	C	D
Spring Dashpot	Mass	Damping Coeff.	Spring Constant	Applied Force
Pendulum	Mass x (Length) ²	Damping x Length	Gravitational Moment	Applied Moment
Series Circuit	Inductance	Resistance	(Capacitance) ⁻¹	Impressed Voltage
Twisted Shaft	Moment of Inertia	Damping	Elastic Shaft Constant	Applied Torque
Diffusion Reaction	Diffusion Constant	Fluid Velocity	Reaction Rate Constant	Source Term

Theoretical Background for Linear 2nd Order DE's

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \quad (1)$$

Note that $g(t)$ represents the “forcing” or “source” terms in (1), whereas $p(t)$ and $q(t)$ represent the physical system. Equation (1) is called **homogeneous** if $g(t) = 0$ for all $t \in I$, and is called **non-homogeneous** if $g(t) \neq 0$ for all $t \in I$.

Definition of a solution to (1), on an interval $t \in I = (\alpha, \beta)$

Example: Verify that t^3 and t are two solutions to the equation: $t^2 y'' - 3ty' + 3y = 0$

The **initial value problem (IVP)** consists of equation (1) and the following **2 conditions**:

$$y(t_0) = y_0 \quad y'(t_0) = y'_0 \quad (2)$$

where y_0 and y'_0 are constants, and $t_0 \in I$.

Why 2 conditions?

Is there always a solution to an IVP?

Theorem 1: Consider the IVP, where $p(t)$, $q(t)$ & $g(t)$ are continuous for $t \in I = (\alpha, \beta)$.

$$\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = g(t) \quad t \in I$$

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad t_0 \in I$$

There is a single solution to this IVP and the solution is defined throughout the time interval I .

Notes:

Principle of Superposition

If y_1 & y_2 are solutions of:

$$\frac{d^2 y_1}{dt^2} + p(t) \frac{dy_1}{dt} + q(t)y_1 = g_1(t) \quad t \in I$$

$$\frac{d^2 y_2}{dt^2} + p(t) \frac{dy_2}{dt} + q(t)y_2 = g_2(t) \quad t \in I$$

then the linear combination: $y = c_1 y_1 + c_2 y_2$, for any values of the constants c_1 & c_2 , satisfies:

$$\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = c_1 g_1(t) + c_2 g_2(t) \quad t \in I$$

Two important special cases:

1. No forcing functions: $g_1(t) = 0$ and $g_2(t) = 0$. Linear combinations of solutions to the homogeneous problem are also solutions to the homogeneous problem
2. Identical forcing functions: $g_1(t) = g_2(t) = g(t)$. See below

Note first that $y_c = y_1 - y_2$ is a solution of the **homogeneous** problem:

Suppose now that $y_p = y_2$ is a particular solution of the **inhomogeneous** equation (1), then any other solution, say $y = y_1$, can be written as:

$$y = y_1 = (y_1 - y_2) + y_2 = y_c + y_p$$

[General solution] = [Complementary solution] + [Particular solution]

To solve (1) we need to:

1. Find the complementary solution y_c , which is general solution to the **homogeneous** problem:

$$\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = 0 \quad (3)$$

2. Find any particular solution y_p of the inhomogeneous problem

Note that y_c , which is the general solution to (3) will contain 2 constants that can be set to satisfy the initial conditions.

Theory of Homogeneous 2nd order linear DE's, i.e. equation (3)

We know that constant multiples of solutions to (3) are also solutions to (3). We have 2 initial conditions and so in some situations will certainly need 2 free constants to satisfy them. Therefore, let's consider two solutions y_1 & y_2 to (3).

1. Satisfying initial conditions at a fixed time: $t_0 \in I$

Suppose y_1 and y_2 are 2 solutions of (3), under what conditions can we find constants c_1 and c_2 so that the linear combination: $c_1y_1 + c_2y_2$, satisfies the initial conditions:

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad t_0 \in I \quad (4)$$

For two differentiable functions, $y_1(t)$ and $y_2(t)$, the **Wronskian determinant** is denoted $W(y_1, y_2)(t)$, and is defined by:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t) \quad (5)$$

Note:

2. Satisfying conditions at all times in I ?

We can show that if $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$, then in fact $W(y_1, y_2)(t) \neq 0$ for all $t \in I$.

3. Fundamental solutions:

A set of 2 solutions, $y_1(t)$ and $y_2(t)$, to equation (3) on the time interval I , is called a **fundamental set of solutions** if $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$.

If $y_1(t)$ & $y_2(t)$ are a set of fundamental solutions to (1) on I , then the function $y(t)$:

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \quad (6)$$

also solves (3), for arbitrary constants c_1 and c_2 , and is called the **general solution**.

Notes:

- Every solution to (1) on I , can be expressed in the form (2), for an appropriate choice of constants c_1 and c_2 .
- Solving **any** linear 2^{nd} order differential equation is reduced to finding 2 solutions, $y_1(t)$ & $y_2(t)$ with non-zero Wronskian
- The general solution is the same as the complementary solution

4. Linear dependence:

Two functions f and g are **linearly dependent** on $I = (\alpha, \beta)$ if there exist constants k_1 and k_2 , not both zero, such that

$$k_1 f(t) + k_2 g(t) = 0 \quad \forall t \in I$$

The functions are **linearly independent** on I if they are not linearly dependent on I .

Note that if f and g are differentiable functions on I :

- If $W(f, g)(t_0) \neq 0$, for $t_0 \in I$, then f and g are linearly independent on I .
- If f and g are linearly dependent on $I = (\alpha, \beta)$, then $W(f, g)(t_0) = 0$ on $I = (\alpha, \beta)$.

To see this:

Example: Consider the linear equation: $t^2 y'' - 2y = 0$

(i) Show that this equation has 2 solutions $y_1(t)$ & $y_2(t)$ of form t^n .

(ii) Show that $y_1(t)$ & $y_2(t)$ are linearly independent.

(iii) Find the solution with initial conditions: $y(1) = 0$, $y'(1) = -3$

Summary for linear homogeneous equations:

- For linear equations with continuous coefficients functions, $p(t)$ & $q(t)$, there exists a unique solution to an IVP.
- Linear combinations of solutions are solutions
- The Wronskian of two solutions is a determinant
- If the Wronskian of 2 solutions is non-zero at a point, then any initial values can be satisfied at that point
- If the Wronskian of 2 solutions is non-zero at a point, then the general solution of the DE is a linear combination of those 2 solutions
- Such a set of solutions is called a **fundamental set of solutions**
- If y_1 and y_2 are two solutions to (3), the following 4 statements are equivalent:
 1. The functions y_1 and y_2 are a fundamental set of solutions on I
 2. The functions y_1 and y_2 are linearly independent on I .
 3. $W(y_1, y_2)(t) \neq 0$ for all $t \in I$.
 4. $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$.

Homogeneous 2nd Order Equations with Constant Coefficients:

Examples: all the physical models that we looked at had form.

$$a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = 0 \quad (7)$$

Can we find solutions to (7) of form $y(t) = e^{rt}$?

Since $e^{rt} \neq 0$, this equation can only have a solution of form $y(t) = e^{rt}$ if the **characteristic equation**:

$$ar^2 + br + c = 0$$

has a solution. This gives the possible values for r :

$$r = r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 cases:

- Two distinct real roots if: $b^2 - 4ac > 0$
- Two distinct complex roots if $b^2 - 4ac < 0$
- Two repeated roots if: $b^2 - 4ac = 0$

Case of 2 real roots: $b^2 - 4ac > 0$

- Both $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are solutions.

- Evaluate the Wronskian:

$$W(e^{r_1 t}, e^{r_2 t})(t) =$$

- Any initial conditions: $y(t_0) = y_0$, $y'(t_0) = y'_0$, can be satisfied by a solution of form:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

with an appropriate choice of c_1 and c_2 .

- The solution: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where c_1 and c_2 are arbitrary constants, is called the **general solution** of (3), in the case that we have 2 real roots of the characteristic equation.

Examples:

1. Solve the IVP: $y'' - 2y' - 2y = 0$: $y(0) = 1$, $y'(0) = -1$

2. Solve the IVP: $y'' + 3y' + 2y = 0$: $y(0) = 1, y'(0) = -1$

Notes on qualitative behaviour:

In general, we could have any combination of real exponents:

- If $0 < r_1 < r_2$, the solution grows exponentially and at large times the term $c_2 e^{r_2 t}$ will dominate.
- If $r_1 < 0 < r_2$ in general, the solution grows exponentially and at large times the term $c_2 e^{r_2 t}$ will dominate. However, there are initial conditions such that $c_2 = 0$, and for these solutions we have $y(t) = c_1 e^{r_1 t}$ which decays to zero, i.e. there could be extreme sensitivity to initial conditions.
- If $r_1 < r_2 < 0$ then both exponents are negative and all solutions decay to zero. The first term decays faster and the decay will be controlled by the term $c_2 e^{r_2 t}$.

Case of 2 distinct complex roots: $b^2 - 4ac < 0$

The roots are given by:

$$r_1 = \lambda + i\mu$$

$$r_2 = \lambda - i\mu$$

$$\lambda = \frac{-b}{2a}, \quad \mu = \frac{\sqrt{4ac - b^2}}{2a}$$

The 2 functions $e^{\lambda t + i\mu t}$ & $e^{\lambda t - i\mu t}$ are solutions to (7). What is the exponential of a complex number?

An aside: Euler's formulae (& the most beautiful equation in the world, $1 + e^{i\pi} = 0$).

From Euler's formulae we see that:

$$e^{(\lambda+i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t} [\cos \mu t + i \sin \mu t]$$

$$e^{(\lambda-i\mu)t} = e^{\lambda t} e^{-i\mu t} = e^{\lambda t} [\cos \mu t - i \sin \mu t]$$

but we do not want complex solutions to (7). Linear combinations of solutions to (7) are solutions, and therefore 2 other solutions to (7) are:

$$y_1(t) = \frac{e^{\lambda t+i\mu t} + e^{\lambda t-i\mu t}}{2} = e^{\lambda t} \cos \mu t$$

$$y_2(t) = \frac{e^{\lambda t+i\mu t} - e^{\lambda t-i\mu t}}{2i} = e^{\lambda t} \sin \mu t$$

Can we satisfy arbitrary initial conditions with a solution of form:

$$y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$$

Examine the Wronskian:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} e^{\lambda t} \cos \mu t & e^{\lambda t} \sin \mu t \\ e^{\lambda t} (\lambda \cos \mu t - \mu \sin \mu t) & e^{\lambda t} (\mu \cos \mu t + \lambda \sin \mu t) \end{vmatrix} \\ &= \mu e^{2\lambda t} \neq 0 \end{aligned}$$

If the roots of the characteristic equation are complex the **general solution** to (7) is given by:

$$y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$$

.

Examples:

1. Solve the IVP: $y'' + 2y' + 5y = 0$: $y(0) = 1, y'(0) = -1$

2. Solve the IVP: $y'' + 9y = 0$: $y(0) = 1, y'(0) = 3$

Notes on qualitative behaviour:

The possibilities with qualitatively different behaviours are:

- Exponentially decaying periodic oscillation: $\lambda < 0$
- Sustained periodic oscillation: $\lambda = 0$
- Exponentially growing periodic oscillation: $\lambda > 0$

Case of a Repeated Root: $r = -(b/2a)$

We know that $y_1(t) = e^{rt}$ is a solution. A second solution is:

$$y_2(t) = te^{rt}$$

To see this, note that:

$$y_2'(t) = rte^{rt} + e^{rt}$$

$$y_2''(t) = r^2te^{rt} + 2re^{rt}$$

and therefore:

$$ay_2'' + by_2' + cy_2 = (ar^2 + br + c)te^{rt} + (2ar + b)e^{rt} = 0$$

since $r = -(b/2a)$ and r satisfies the characteristic equation. Can we satisfy arbitrary initial conditions with a solution of form:

$$y(t) = c_1e^{rt} + c_2te^{rt}$$

Check the Wronskian:

$$W(e^{rt}, te^{rt})(t) = \begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & (rt+1)e^{rt} \end{vmatrix} = e^{2rt} \neq 0$$

If the characteristic equation has repeated roots the **general solution** of (7) is:

$$y(t) = c_1e^{rt} + c_2te^{rt}$$

.

Examples:

1. Solve the IVP: $y'' - 6y' + 9y = 0$: $y(0) = -1, y'(0) = 1$

2. Solve the IVP: $y'' + 6y' + 9y = 0$: $y(0) = -1, y'(0) = 1$

Notes on qualitative behaviour for repeated roots:

- Decaying if: $r < 0$
- Growing if : $r > 0$
- Note that growth to $\pm \infty$ is faster than exponential and decay to zero is slower than exponential

Summary:

For linear equations with constant coefficients, we do the following:

1. Write down the characteristic equation and find the roots
2. Write down the general solution according to the roots
 - Real & distinct $\Rightarrow y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
 - Complex and distinct $\Rightarrow y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$
 - Real and repeated $\Rightarrow y(t) = c_1 e^{rt} + c_2 t e^{rt}$
3. Fit the constants c_1 and c_2 to the initial conditions

Inhomogeneous Linear 2nd Order Differential Equations

Equations like:

$$\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = g(t) \quad (1)$$

with $g(t) \neq 0$ for all $t \in I$, (i.e. $g(t) = 0$ is possible for some $t \in I$, but not every $t \in I$).

Recall the **general solution** to the inhomogeneous problem (1) is:

$$y(t) = y_c(t) + y_p(t)$$

where $y_c(t)$ is the general solution to the homogeneous problem:

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

and $y_p(t)$ is any solution of (1).

Constant coefficient equations: method of undetermined coefficients

Method for finding $y_p(t)$ for equations like:

$$a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = g(t)$$

Suggestions for guessing: (exactly as with 1st order equations)

Forcing function $g(t)$	Guess for $y_p(t)$
$\sum_{j=0}^N b_j t^j$: with constants b_j	$\sum_{j=0}^N c_j t^j$
e^{bt}	Ae^{bt}
$\sin bt$ or $\cos bt$	$A \cos bt + B \sin bt$
$e^{ct} \sin bt$ or $e^{ct} \cos bt$	$Ae^{ct} \cos bt + Be^{ct} \sin bt$
Multiplicative combinations of above	Multiplicative combinations of above
Additive combinations of above	Additive combinations of above

Procedure:

- Assume the form of function for $y_p(t)$, with arbitrary constant coefficients.
- Substitute in the equation
- Determine the coefficients

Example 1: Find a particular solution of the following.

a) $y'' - 3y' - 4y = 3e^{2t}$

b) $y'' - 3y' - 4y = 2\sin t$

Example 2: Solve the IVP: $y'' - 3y' - 4y = -8e^t \cos 2t$,
 $y(0) = y'(0) = 1$

Mathematical resonance & undetermined coefficients:

Occurs when $g(t)$ takes the same form as the general solution of the homogeneous problem.

Trick: multiply your usual guess for $y_p(t)$ by t^k where k is large enough to ensure that your guess is not of the same form as $y_c(t)$.

Example 3: Find the general solution of: $y'' + 5y' + 4y = e^{-4t}$

Example 4: Express the solution of the IVP

$$y'' + 4y = \sin \omega t, \quad y(0) = 1, \quad y'(0) = 0$$

as a function of ω and t . For what values of ω does the IVP have solutions that become unbounded as $t \rightarrow \infty$?

Example 5: Find the general solution of: $y'' + 4y' + 4y = e^{-2t}$

Method of reduction of order:

General method that works even with equations like:

$$\frac{d^2 y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \quad (1)$$

and/or constant coefficients with complicated functions $g(t)$. The method will also find a 2nd solution to a homogeneous problem if you have already found one solution.

Method:

1. Suppose a single solution $y_1(t)$ of the homogeneous equation is known, with $y_1(t) \neq 0$ on I
2. Look for a solution $y_p(t)$ of form:

$$y_p(t) = u(t)y_1(t)$$

3. where $u(t)$ is a function to be determined from (1).
4. Differentiating the function $y_p(t)$:

$$y_p'(t) = u(t)y_1'(t) + u'(t)y_1(t)$$

$$y_p''(t) = u(t)y_1''(t) + 2u'(t)y_1'(t) + u''(t)y_1(t)$$

5. Substitute into (1):

$$u(t)y_1''(t) + 2u'(t)y_1'(t) + u''(t)y_1(t) + p(t)[u(t)y_1'(t) + u'(t)y_1(t)] + q(t)u(t)y_1(t) = g(t)$$

and after simplifying

$$y_1(t)u''(t) + [2y_1'(t) + p(t)y_1(t)]u'(t) = g(t)$$

6. This is a first order linear equation for $u'(t)$, which we solve using integrating factor method.

$$\left[u'' + \frac{(2y_1' + py_1)}{y_1} u' \right] = \frac{g(t)}{y_1(t)}$$

Integrating factor is $y_1^2(t)e^{\int p(s) ds}$

7. Finally the formula for $u'(t)$ is:

$$u'(t) = \frac{\int g(\tilde{t})y_1(\tilde{t})e^{\int p(s) ds} d\tilde{t} + C}{y_1^2(t)e^{\int p(s) ds}}$$

8. Integrate for $u(t)$ and multiply by $y_1(t)$ to get $y_p(t)$

Comments:

- Final integral expression is complex - learn the constructive method, not the formulae!
- By setting $g(t) = 0$ in the above we get a second solution to the homogeneous problem.

Example 1: Find a particular solution of:

$$ty'' - 2ty' + 2y = 4t^2 \quad t > 0$$

given that $y_1(t) = t$ is a solution to the homogeneous problem.

Example 2: Consider the linear equation: $t^2 y'' - 2y = 3t^2 - 1$

- i. Show that the homogeneous equation has a solution of form t^n .
- ii. Find a particular solution.

Spring Oscillations

Derivation:

- Let L be the natural length of the spring, m the mass of the weight and $-f(t)$ an applied force.
- Let the length of the spring be denoted $l(t)$
- From Newton's second law: Force = Mass x Acceleration

$$m \frac{d^2 l}{dt^2} = mg + f_d(t) + f_s(t) - f(t)$$

where g is the gravitational acceleration, $f_d(t)$ is the force due to damping and $f_s(t)$ is the force that the spring exerts.

- Hooke's law gives:

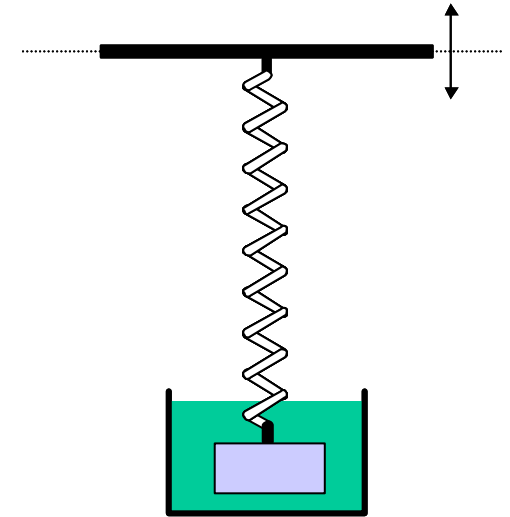
$$f_s(t) = -k[l(t) - L]$$

where k is the **spring constant**.

- The damping force $f_d(t)$, is due to friction with the fluid and opposes the motion of the weight. It is assumed that friction is proportional to the speed of the weight:

$$f_d(t) = -\beta \frac{dl}{dt}$$

where β is the **damping coefficient**.



- Consider an equilibrium position, with no external forcing and no movement of the spring. The equation of motion is:

$$0 = mg - k[L^* - L]$$

where $l(t) = L^*$ is the **equilibrium length**, which we write as:

$$L^* = L + \Delta L$$

i.e. $\Delta L = mg/k$ is the **equilibrium extension** of the spring.

- Putting this together, we have:

$$m \frac{d^2 l}{dt^2} = mg - \beta \frac{dl}{dt} - k[l(t) - L] - f(t)$$

- Now define $x(t)$ by:

$$l(t) = L + \Delta L - x(t)$$

i.e. $x(t)$ measures the distance above the equilibrium position.

- Finally we get:

$$m \frac{d^2 x}{dt^2} = -kx - \beta \frac{dx}{dt} + f(t)$$

Simple Harmonic Motion

This refers to situations with no damping and no forcing, i.e.

$$m \frac{d^2 x}{dt^2} + kx = 0 \quad (1)$$

General solution:

The parameter ω_0 is the **natural frequency** of (1). Periodic motion with period: $T = 2\pi / \omega_0$

Amplitude-phase form of solution:

Idea: write our solution in the form: $y(t) = R \cos(\omega_0 t - \phi)$, where R is the **amplitude** and ϕ is the **phase angle**. By convention, $-\pi \leq \phi \leq \pi$.

Example: The natural length of a spring is $1m$. A mass is attached to the end and the length increases to $1.02m$, when in equilibrium. The mass is initially displaced downwards by $1cm$ from the equilibrium position and is released with upwards velocity $14cm/s$

a) Find the displacement for $t > 0$

b) Find the natural frequency, period, amplitude and phase angle of the motion.

Undamped Forced Oscillations:

What happens close to the natural frequency of an oscillator? Consider the IVP:

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t : \quad y(0) = 0, \quad y'(0) = 0,$$

The solution is:
$$y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} [\cos \omega t - \cos \omega_0 t]$$

Remembering that:

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

we write: $\omega t = A - B$ & $\omega_0 t = A + B$, and find

$$A = \frac{(\omega_0 + \omega)t}{2} \quad \text{and} \quad B = \frac{(\omega_0 - \omega)t}{2}$$

Putting this together, the solution is:
$$y(t) = R(t) \sin \left[\frac{(\omega_0 + \omega)t}{2} \right]$$

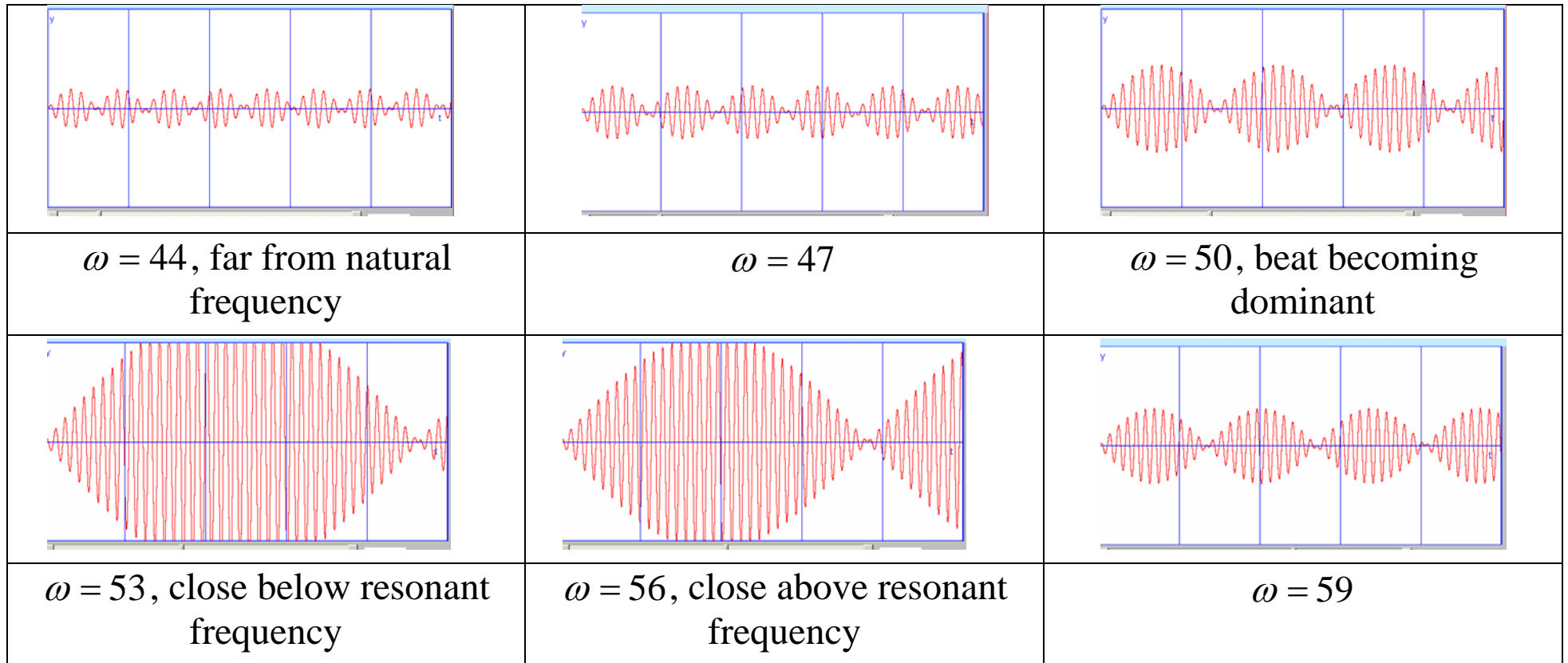
where

$$R(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \left[\frac{(\omega_0 - \omega)t}{2} \right]$$

Notes:

- The term $R(t)$ gives the amplitude of the solution
- For ω far from ω_0 , there may be virtually no interaction between $R(t)$ and $\sin \left[\frac{(\omega_0 + \omega)t}{2} \right]$
- For ω close to ω_0 , the frequency of $\sin \left[\frac{(\omega_0 + \omega)t}{2} \right]$ approaches the natural frequency. $R(t)$ becomes large and the period of $R(t)$ becomes very long: $T = 4\pi / |\omega - \omega_0|$.

Example: $\omega_0 = 54$



Free oscillations with damping:

$$m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx = 0$$

- What happens for “small” and “large” β ?
- What do we mean by “small” and “large” β ?

Characteristic equation is: $mr^2 + \beta r + k = 0$, with roots:

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

3 different cases:

- $\beta^2 < 4km$: roots are complex – **underdamped** motion
- $\beta^2 > 4km$: roots are real – **overdamped** motion
- $\beta^2 = 4km$: repeated roots – **critically damped** motion

For $\beta > 0$ the real part of $r_{1,2}$ is always negative. Therefore, we always have exponential decay of the solutions as $t \rightarrow \infty$.

Positive, negative, linear and nonlinear damping:

We call systems with $\beta > 0$, **positively damped**, & systems with $\beta < 0$, **negatively damped**. If β is not constant, but a function of x and/or x' , this is referred to as **nonlinear** damping. We consider only linear positively damped systems.

Underdamped motion:

General solution is: $x(t) = e^{-\frac{\beta}{2m}t} [c_1 \cos \omega_1 t + c_2 \sin \omega_1 t]$, where: $\omega_1 = \sqrt{k/m - (\beta/2m)^2}$

We can write this in amplitude phase form:

- $e^{-\frac{\beta}{2m}t} R$ is the **time-varying amplitude** (or amplitude)
- ω_1 is the **quasi-frequency** of the motion
- $T = 2\pi / \omega_1$ is the **quasi-period** of the motion

Overdamped motion:

General solution is: $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, with $r_1 < r_2 < 0$.

Critically damped motion:

General solution is:

$$x(t) = c_1 e^{rt} + c_2 t e^{rt}$$

with $r < 0$.

Example 1: A mass of 64g stretches a spring 6cm in equilibrium and a dashpot provides a linear damping force to the system.

- a) Find the equation of motion for the displacement $x(t)$ of the mass, above its equilibrium.
- b) Determine the value of damping constant β for which the system is critically damped.
- c) Find the displacement $x(t)$ if the system is critically damped and the mass is released from position $x(0) = 3\text{cm}$ above the equilibrium, with zero velocity. Express the answer in amplitude-phase form.
- d) How large must the damping constant be in order to ensure that the mass never oscillates less than 3mm below the equilibrium, assuming an underdamped system?

Forced damped oscillations:

$$m \frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} + kx = F_0 \cos \omega t$$

General solution will be of form:

$$x(t) = x_c(t) + x_p(t)$$

Where $x_c(t)$ has one of the following forms:

$$x_c(t) = \begin{cases} e^{-\frac{\beta}{2m}t} [c_1 \cos \omega_1 t + c_2 \sin \omega_1 t] & \text{if } \beta^2 < 4km \\ c_1 e^{r_1 t} + c_2 e^{r_2 t} & \text{if } \beta^2 > 4km \\ c_1 e^{rt} + c_2 t e^{rt} & \text{if } \beta^2 = 4km \end{cases}$$

Using method of undetermined coefficients, $x_p(t)$ is of form:

$$x_p(t) = A \cos \omega t + B \sin \omega t$$

What happens to $x(t)$ as $t \rightarrow \infty$?

- We call $x_c(t)$ the transient component of the solution
- We call $x_p(t)$ the steady state component of the solution

Example 2: (How fast should you drive along bumpy roads?) Assuming that m , β , k and F_0 are held constant, what value of forcing frequency ω produces the largest amplitude steady state response? The smallest steady state response?