

Why study ODE's and PDE's?

- Core course and a prerequisite to continue with other courses
- Intellectually: to understand and solve differential equations
- Ordinary differential equations model dynamical systems with many applications:
 - financial
 - ecological & biological systems
 - chemical reactions, mixing
 - physical/mechanical/electrical
- Partial differential equations are fundamental to
 - fluid mechanics
 - heat transfer
 - solid mechanics
 - electrical engineering
 - magnetism, relativity, planetary motion....

Basis of many technical engineering jobs using e.g. CFD or FEM software.

- New developments, (e.g. chaos, stochastic PDE's for derivative modelling).
- Historical perspective

Historical Perspective:

- Isaac Newton (1642-1727)
 - Newtonian mechanics
 - Fundamental results of calculus
 - Classified DE's & some solution methods
 - 1687 began to publish his results
- Gottfried Leibniz (1646-1716)
 - Fundamental results of calculus
 - Introduced notation
 - Many solution methods for DE's: e.g. separation of variables
 - 1684 began to publish his results
- 1700's
 - Bernoulli's, Euler, Lagrange
- 1800's
 - PDE's & more fundamental questions
 - Laplace, d'Alembert, Maxwell, Cauchy, etc,

All the “great men of science” had their names inscribed on the Eiffel tower when constructed



Their achievements are largely associated with deriving and solving differential equations that model practical problems

Basic terminology

- Differential equations:
 - Ordinary differential equations (ODE's)
 - Partial differential equation (PDE's)
- System of differential equations
- Order of a differential equation
- Solution of a differential equation
- Classifying differential equations
 - Linear: $y' + p(t)y = g(t):$ $y(t_0) = y_0$
 - Nonlinear: $y' = f(t, y):$ $y(t_0) = y_0$
- Initial Conditions

Direction fields: $y' = f(t, y)$

With a computer

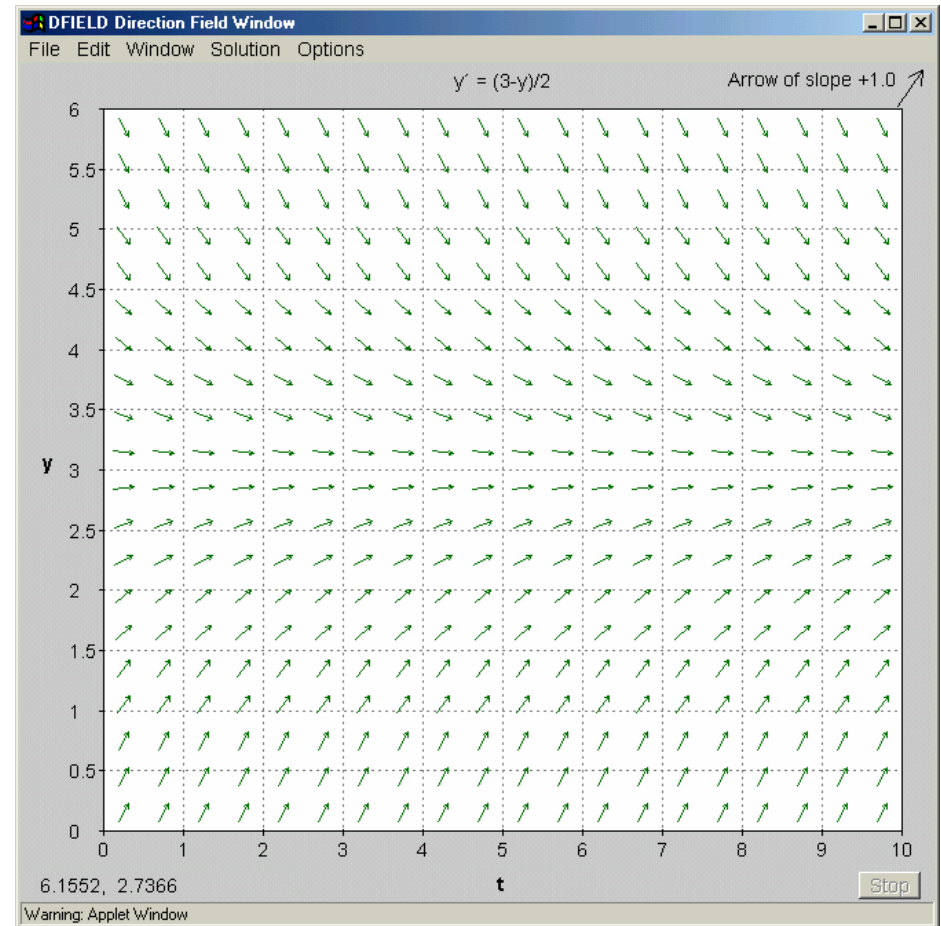
- Make a graph of t against y
- Make a mesh of points (t_i, y_j)
- Draw an arrow in the direction of the derivative $y'(t)$ at each point (t_i, y_j)
- See e.g. DFIELD, for use with Matlab:
<http://math.rice.edu/~dfield/>

Without a computer

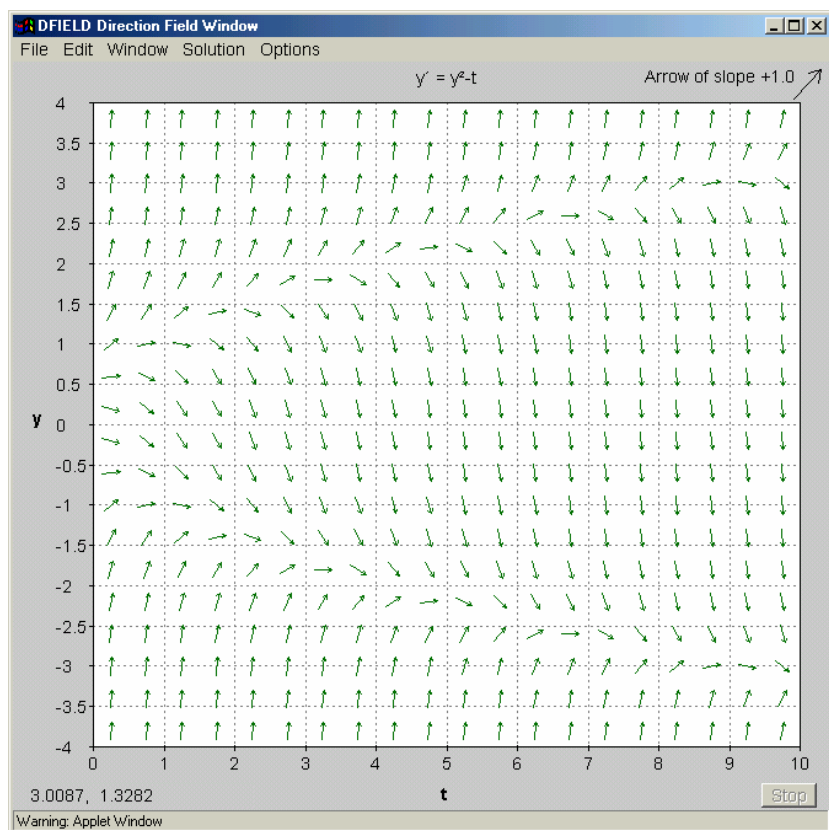
- Sketch the **isocline(s)** where $f(t, y) = 0$.
- On each side of an isocline $y(t)$ will be either increasing or decreasing. Usually (but not always) the slope of $y(t)$ changes sign across each isocline.
- Look in each region separated by nullclines at how slope changes for large values of (t, y) .

Example: Sketch direction field for:

$$y' = -3y + t$$



Example: Sketch the direction field for the ODE: $y' = y - t^2$



Homogeneous Constant Coefficient Equations

Simplest differential equations possible.

Equations like:

$$y' + ay = 0 \tag{1}$$

where a is a constant: $a > 0$ or $a < 0$.

General solution of (1):

Initial condition for (1):

$$y(t_0) = y_0 \tag{2}$$

Initial value problem (IVP) consists of (1) & (2). Solution of IVP is solution of (1) that satisfies (2).

Solution of the IVP:

Dependence on initial conditions:

Equilibrium solution:

Inhomogeneous Constant Coefficient Equations

Equations like:

$$y' + ay = g(t) \quad (3)$$

where a is a constant: $a > 0$ or $a < 0$ and $g(t)$ is a given function of t . Examples:

1. $y' + y = t$
2. $y' - 5y = e^t$
3. $y' + ay = \sin bt$

Superposition principle:

Suppose that $y_1(t)$ and $y_2(t)$ are two solutions to (3). Then it is obvious that the function $y_c(t)$:

$$y_c(t) = y_1(t) - y_2(t)$$

will satisfy the **homogeneous** equation (1), i.e. $y' + ay = 0$.

Suppose we find any **particular solution**, $y_p(t)$, to (3). The general solution to (3) will be of form:

$$y(t) = y_c(t) + y_p(t) \quad (4)$$

We call the function $y_c(t)$ the **complementary solution**, which is equivalent to the general solution of the homogeneous problem (1). Note, $y_c(t)$ contains a constant that can be used to fit initial conditions.

Method of Undetermined Coefficients – one method of finding $y_p(t)$?

- A method that is quick (also called guessing!)
- Works only for constant coefficient equations and certain $g(t)$.

Examples:

1. Find the general solution of: $y' + y = t$

2. Solve the IVP: $y' - 5y = e^t$ $y(0) = 2$

3. Find the general solution of: $y' + ay = \sin bt$

Suggestions for guessing:

Forcing function $g(t)$	Guess for $y_p(t)$
$\sum_{j=0}^N b_j t^j$: with constants b_j	$\sum_{j=0}^N c_j t^j$
e^{bt}	Ae^{bt}
$\sin bt$ or $\cos bt$	$A \cos bt + B \sin bt$
$e^{ct} \sin bt$ or $e^{ct} \cos bt$	$Ae^{ct} \cos bt + Be^{ct} \sin bt$
Additive combinations of above	Additive combinations of above
Multiplicative combinations of above	Multiplicative combinations of above
e^{-at}	
Any other functional form	

Basic method to solve an inhomogeneous IVP consisting of equations (3) and an initial condition

1. Find general solution of homogeneous problem: $y_c(t) = Ce^{-at}$
2. Consider the form of $g(t)$, guess the correct form for $y_p(t)$.
3. Substitute $y_p(t)$ into (3) to determine the constants in $y_p(t)$.
4. General solution is $y_c(t) + y_p(t)$
5. Evaluate $y_c(t) + y_p(t)$ at $t = t_0$ to find constant C

“Mathematically resonant” solutions

Method of undetermined coefficients does not work quite so simply if $g(t)$ is a solution of the homogeneous problem, i.e.

$$y' + ay = ce^{-at} : a \neq 0, c \neq 0$$

Problem:

Solution: multiply by t : seek a particular solution of form:

$$y_p(t) = Ate^{-at}$$

Examples:

4. Solve the IVP: $y' - 5y = -e^{5t}$, $y(0) = y_0$

5. Solve the following IVP and compare the behaviour of the solution with the general behaviour of solutions to the homogeneous equation.

$$y' + y = e^{-t}, \quad y(0) = 1$$

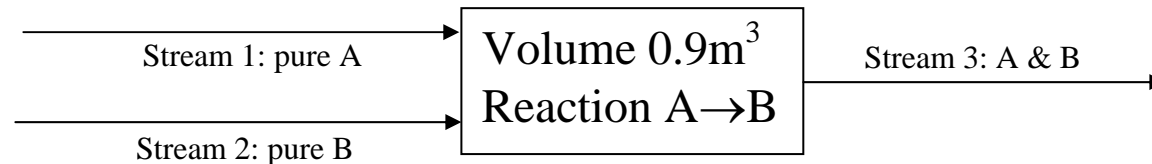
Applications that lead to 1st order linear differential equations

1. “Mixing”: unsteady material and energy balances

Basic principle is a control volume analysis:

$$[\text{Rate ACC}] = [\text{Rate IN}] - [\text{Rate OUT}] + [\text{Rate GEN}] - [\text{Rate CON}]$$

Example: Consider an isothermal continuous stirred tank reactor (CSTR) as shown, in which an irreversible reaction takes place that converts substance A to substance B at rate $5[A]$ kmol/(m³.h).



Here $[A] = [\text{kmol/h of A}] / [\text{m}^3/\text{h through CSTR}] = \text{concentration of A}$.

Initially the system is in a steady state with 10kmol/h of A flowing through stream 1 and 30kmol/h of B flowing through stream 2. The material volume in the CSTR stays constant at 0.9m³, temperature is kept constant and the outflow in stream 3 has the same composition as that inside the CSTR. Substance A has specific mass 40kg/kmol and density 900kg/m³. Substance B has specific mass 80kg/kmol and density 900kg/m³. At time $t=0$, the streams 1&2 are changed to 50kmol/h of A and 20kmol/h of B.

Calculate the outflow concentration $[A]$ as a function of time.

2. Newton's law of heating:

A solid body loses heat to its surrounding medium. The rate of heat loss is observed to be proportional to the temperature difference between the solid body and surrounding medium.

Typically:

$$mc_p \frac{dT}{dt} = -hA(T - T_a)$$

models the temperature $T(t)$, where m , c_p , h , A and T_a denote respectively the mass of the body, specific heat of the material, heat transfer coefficient, surface area of the body and temperature of the surrounding medium. Alternatively:

$$\frac{dT}{dt} = -k(T - T_a)$$

and k is fitted from observation. The **relaxation time**, ($= 1/k$) is the time taken for the temperature difference $(T - T_a)$ to decay to e^{-1} of its initial value.

Example: During a murder investigation, a corpse was found by Inspector Clouseau at exactly 8:00p.m. Being alert, he measures the temperature of the body and finds it to be 70°F . Two hours later, Inspector Clouseau again measures the temperature of the corpse and finds it to be 60°F .

- a) If the room temperature is 50°F , and we assume that Newton's law of cooling applies, when did the murder occur? (Assume that the temperature of the body at the time of the murder was 98.6°F).
- b) What is the relaxation time of the corpse?

3. Other applications: any growth/decay process with rate proportional to the existing population:

- a) Radioactive decay
- b) Savings, investments and debts
- c) Unbounded population growth, e.g. bacteria, rabbits,

Integrating Factor Method

General method for equations like:

$$y' + p(t)y = g(t) \quad (5)$$

Idea: we try to find a function $\mu(t)$ such that:

$$\mu(t)y' + \mu(t)p(t)y = \frac{d}{dt}[\mu(t)y] \quad (6)$$

The function $\mu(t)$ is called the **integrating factor**. If we can find $\mu(t)$, it follows that:

$$\frac{d}{dt}[\mu(t)y(t)] = \mu(t)g(t) \quad (7)$$

which we integrate:

$$\mu(t)y(t) = \int^t \mu(s)g(s)ds + C$$

and finally get the **general solution** of (5):

$$y(t) = \frac{\int^t \mu(s)g(s)ds + C}{\mu(t)} \quad (8)$$

How do we find $\mu(t)$?

$$\begin{aligned}\mu y' + \mu p y &= \frac{d}{dt}[\mu y] = \mu y' + \mu' y \Leftrightarrow \mu p = \mu' \\ &\Leftrightarrow \frac{d}{dt}[\ln \mu] = p\end{aligned}$$

Therefore we have finally that the integrating factor is:

$$\mu(t) = \exp\left(\int^t p(s) ds\right) \quad (9)$$

Check that this satisfies (6):

Note that the general solution (8) we can write as: $y(t) = y_p(t) + y_c(t)$, where

$$y_p(t) = \frac{\int^t \mu(s) g(s) ds}{\mu(t)} \quad \text{and} \quad y_c(t) = \frac{C}{\mu(t)}$$

and $y_p(t)$ is a particular solution to (5) and $y_c(t)$ is the general solution of the homogeneous part:

$$y' + p(t)y = 0$$

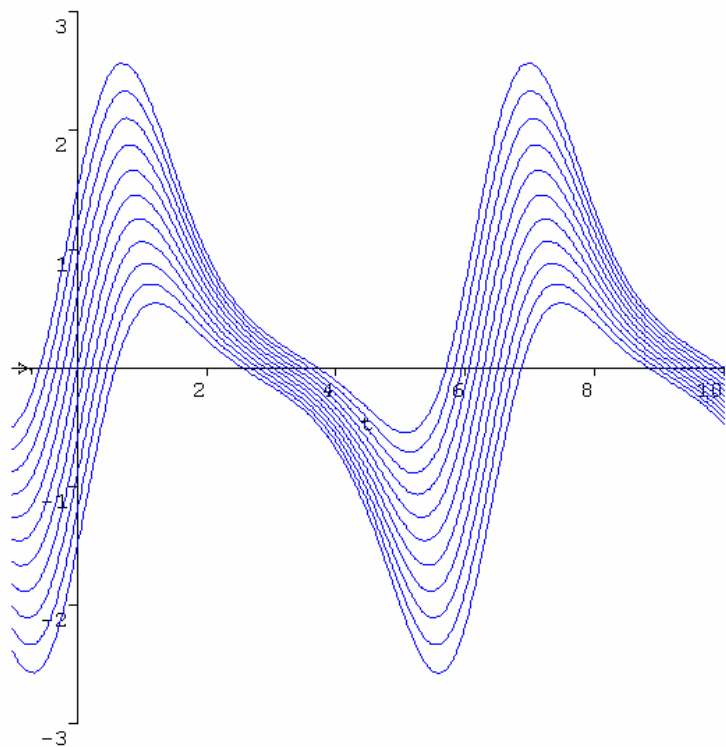
Examples:

1. Find the general solution of: $y' + y = t$

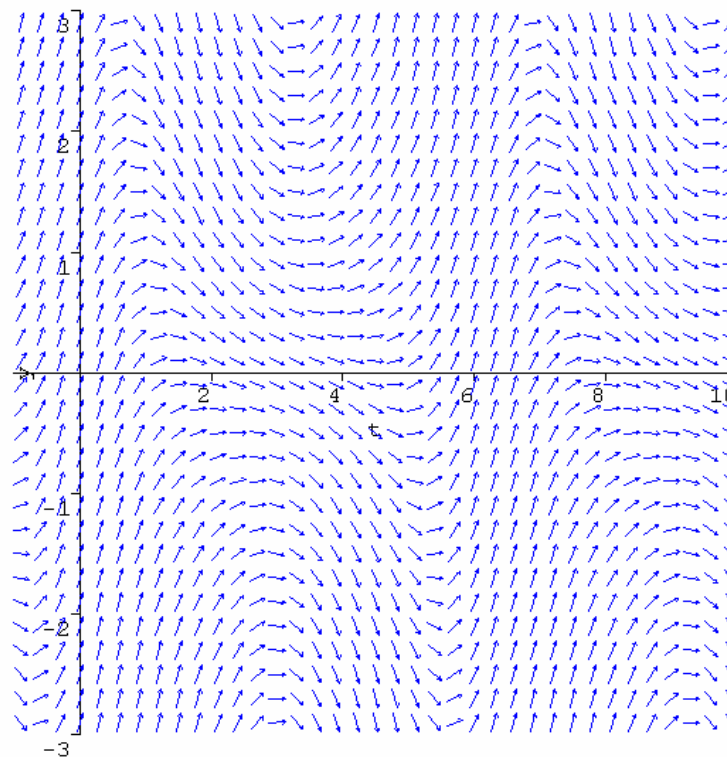
2. Solve the IVP: $y' - 5y = -e^{5t}$, $y(0) = 2$

$$3. \frac{y'}{\cos t} + y \tan t = e^{\cos t}$$

Integral curves: $y(t) = [C + \sin t]e^{\cos t}$



Direction field



General steps for using the integrating factor for an IVP:

1. Put equation in the correct form: $y' + p(t)y = g(t)$

2. Integrate to find $\mu(t) = \exp\left(\int^t p(s)ds\right)$

3. Multiply equation through: $\frac{d}{dt}[\mu(t)y(t)] = \mu(t)g(t)$

4. Integrate to get general solution: $y(t) = \frac{\int^t \mu(s)g(s)ds + C}{\mu(t)}$

5. Find initial condition by fitting the constant C

Note: Step 3 above reduces the problem to a single integration. In a sense this means that looking for the variable $y(t)$ is perhaps not the most sensible way to view a first order linear ODE. For some physical applications, we can readily see a more “natural” variable.

Example: slow flow of an incompressible fluid through a channel with varying permeable walls (blood flow through a narrow artery).

Equations with Non-smooth Data Functions

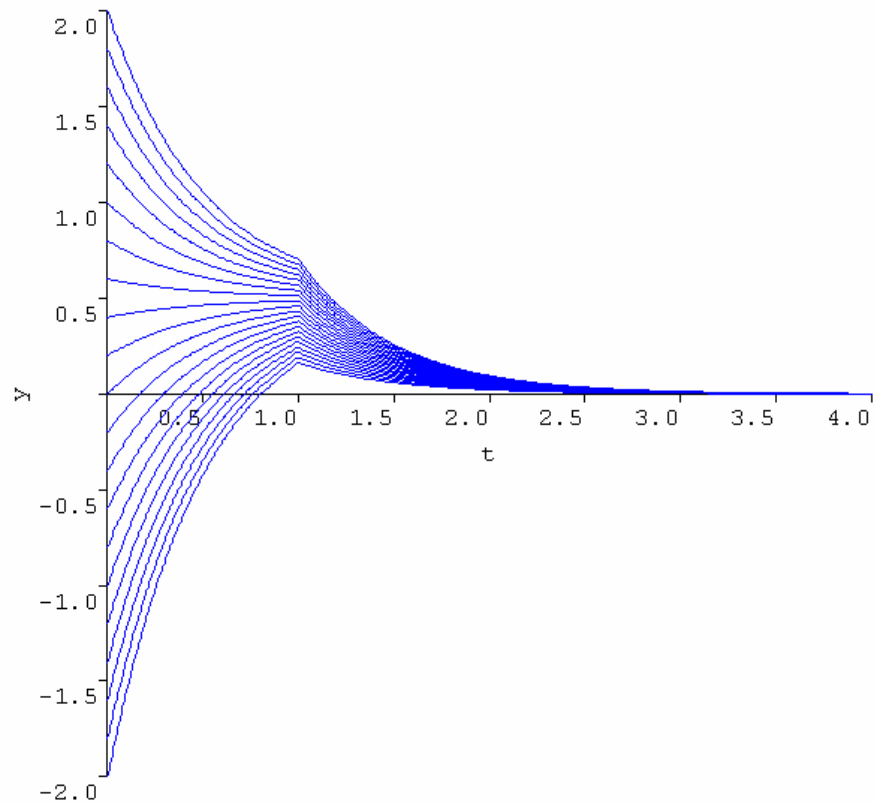
Frequently a solution can be pieced together at the points of discontinuity.

Example 1: Solve the following IVP:

$$y' + 2y = g(t): \quad y(0) = a$$

where

$$g(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & 1 < t \end{cases}$$



Example 2: Suppose that you buy a cup of coffee for Dr Frigaard at time 7.40am, at which time the temperature of the coffee is 70C. After spending 10 minutes in the warm coffee house (at 25C), you measure the temperature of the coffee and find that it has dropped to 65C. You walk quickly to your favourite (Math 256) class, through a blizzard at -10C , and arrive 10 minutes later. How cold is the coffee now & should you give it to Dr Frigaard?