Ward Identities and a Perturbative Analysis of a U(1) Goldstone Boson in a Many Fermion System

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Abstract. We derive Ward identities, to all orders of perturbation theory, in many Fermion systems with short range interactions and broken $U(1)$ number symmetry. They imply that the system is superrenormalizable in the Goldstone boson regime.

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§I Introduction

We consider a many Fermion system, in dimension $d \ge 2$, formally characterized by the (renormalized) generating functional

$$\mathcal{S}(\phi,\bar\phi) \;=\; \log \tfrac{1}{\mathcal{Z}} \int e^{[\bar\phi\psi]+[\bar\psi\phi]}\; e^{\mathcal{A}(\psi,\bar\psi)} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma} \tag{I.1}$$

for the connected Euclidean Green's functions, where the action

$$\mathcal{A}(\psi,\bar\psi) \;=\; -\mathcal{V}(\psi,\bar\psi) \;-\; \delta\mu(\lambda,\mu) \int đk\, \bar\psi_k\psi_k \;-\; \int đk\; \big(ik_0 e(\mathbf{k})\big)\bar\psi_k\psi_k \tag{1.2}$$

and

$$\begin{aligned}\mathcal{V}(\psi,\bar\psi) \;&=\; \tfrac{\lambda}{2} \int \prod_{i=1}^{4} đk_i\; (2\pi)^{d+1}\delta(k_1{+}k_2{-}k_3{-}k_4)\; \bar\psi_{k_1}\psi_{k_3}\,\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle\, \bar\psi_{k_2}\psi_{k_4} \\ &=\; \tfrac{\lambda}{2} \sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \int \prod_{i=1}^{4} đk_i\; (2\pi)^{d+1}\delta(k_1{+}k_2{-}k_3{-}k_4)\,\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle\, \bar\psi_{k_1,\sigma}\bar\psi_{k_2,\tau}\psi_{k_4,\tau}\psi_{k_3,\sigma}\end{aligned} \tag{I.3}$$

In these expressions, the internal and external electron fields

$$\psi(\xi) = \begin{pmatrix}\psi(\xi,\uparrow)\\ \psi(\xi,\downarrow)\end{pmatrix} \qquad\qquad \phi(\xi) = \begin{pmatrix}\phi(\xi,\uparrow)\\ \phi(\xi,\downarrow)\end{pmatrix}$$

$$\bar\psi(\xi) = (\,\bar\psi(\xi,\uparrow)\quad \bar\psi(\xi,\downarrow)\,) \qquad\qquad \bar\phi(\xi) = (\,\bar\phi(\xi,\uparrow)\quad \bar\phi(\xi,\downarrow)\,)$$

where $\psi(\xi,\sigma), \phi(\xi,\sigma), \bar\psi(\xi,\sigma)$ and $\bar\phi(\xi,\sigma)$, $\xi=(t,\mathbf{x})\in \mathbb{R}\times\mathbb{R}^d$, $\sigma\in\{\uparrow,\downarrow\}$, are generators of an infinite dimensional Grassmann algebra over $\mathbb{C}$ and

$$\begin{aligned} k \;&=\; (k_0,\mathbf{k}) \;\in\; \mathbb{R}\times\mathbb{R}^d & đk \;&=\; \tfrac{dk_0}{2\pi}\; đ\mathbf{k} \;=\; \tfrac{d^{d+1}k}{(2\pi)^{d+1}} \\ \psi_{k,\sigma} \;&=\; \textstyle\int d\xi\, e^{i\langle k,\xi\rangle_-}\,\psi(\xi,\sigma) & \langle k,\xi\rangle_- \;&=\; -k_0 t + \langle \mathbf{k},\mathbf{x}\rangle \\ e(\mathbf{k}) \;&=\; \tfrac{\mathbf{k}^2}{2\mathbf{m}} - \mu \end{aligned}$$

When we wish to make explicit the dependence on the chemical potential μ, we expand the notation $e(\mathbf{k})$ to $e(\mathbf{k},\mu)$. For a function f

$$[f] \;=\; \int d\xi\; f(\xi)$$

so that, in particular

$$[\bar\phi\psi] \;=\; \int d\xi\; \bar\phi(\xi)\psi(\xi) \;=\; \sum_{\sigma\in\{\uparrow,\downarrow\}} \int d\xi\; \bar\phi(\xi,\sigma)\psi(\xi,\sigma)$$

We assume that the interaction $\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle$ is real and in addition rotation, reflection and time reversal invariant. Precisely,

$$\begin{aligned}\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle \;&=\; \langle Rk_1,Rk_2|\mathrm{V}|Rk_3,Rk_4\rangle \quad \text{for all } R\in O(d) \\ \langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle \;&=\; \langle Tk_1,Tk_2|\mathrm{V}|Tk_3,Tk_4\rangle\end{aligned} \tag{S1}$$

where $Rk = (k_0, R\mathbf{k})$, $Tk = (-k_0, \mathbf{k})$. We also assume

$$\begin{aligned} \big\langle k_1, k_2 \big| V \big| k_3, k_4 \big\rangle &= \big\langle -k_3, k_2 \big| V \big| -k_1, k_4 \big\rangle \\ &= \big\langle k_1, -k_4 \big| V \big| k_3, -k_2 \big\rangle \end{aligned} \tag{S2}$$

and that $\big\langle k_1, k_2 \big| V \big| k_3, k_4 \big\rangle$ is short range.

The counterterm $-\delta\mu(\lambda, \mu) \int đk\, \bar\psi_k \psi_k$ in the action renormalizes the radius of the Fermi surface. It is discussed in the next section. Finally, the denominator $\mathcal{Z}$ is chosen so that

$$\mathcal{S}(0,0) = 0$$

One of our long term goals, stated precisely near the beginning of the next section, is to give a rigorous proof that the standard model

$$\langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle = \widetilde{V}(\mathbf{k}_1 - \mathbf{k}_3) - \alpha^2 \theta\big(\omega_D - \omega(\mathbf{k}_1 - \mathbf{k}_3)\big)^2 \frac{\omega(\mathbf{k}_1 - \mathbf{k}_3)^2}{(k_1 - k_3)_0^2 + \omega(\mathbf{k}_1 - \mathbf{k}_3)^2} \tag{I.4}$$

for an interacting system of electrons and jellium phonons has a superconducting ground state at sufficiently low temperature. To explain why Ward identities are required to prove that the condensate of Cooper pairs is stable, we briefly review the overall strategy.

We investigate the long range behavior of correlation functions at low temperature using a renormalization group analysis near the Fermi surface [5,6]. This entails slicing the free propagator around its singularity on the Fermi sphere. The renormalization group resums the dominant graphs of the theory to create an effective slice-dependent interaction.

The natural scales correspond to finer and finer shells around the Fermi surface. Fix a constant $M > 1$. For each $j = 0, -1, -2, \ldots$ the j-th slice contains all momenta in a shell of thickness M^j a distance M^j from the singular locus $\big\{\, k \in \mathbb{R}^{d+1} \;\big|\; k_0 = 0,\ |\mathbf{k}| = \sqrt{2\mathbf{m}\mu}\,\big\}$. The propagator for the j-th slice is

$$C^j(\xi_1, \xi_2) = \delta_{\sigma_1, \sigma_2} \int đk \frac{e^{i\langle k, \xi_1 - \xi_2 \rangle_-}}{ik_0 - e(\mathbf{k})} 1_j\big(k_0^2 + e(\mathbf{k})^2\big)$$

where $1_j\big(k_0^2 + e(\mathbf{k})^2\big)$ is the characteristic function for the set $M^j \le |ik_0 - e(\mathbf{k})| < M^{j+1}$. For simplicity, we have introduced a sharp partition of unity even though a smooth one is required for a complete, technically correct analysis [6, II.1]. Summing over $j \le 0$, we obtain the full infrared propagator $C(\xi_1, \xi_2) = \sum_{j \le 0} C_j(\xi_1, \xi_2)$.

These shells induce an infrared renormalization group flow. It was shown in [6] that the important part of (I.3) comes from the reduced interaction $\lambda \langle s', -s' | \mathrm{V} | t', -t' \rangle$. Expanding in spherical harmonics

$$\lambda \langle s', -s' | \mathrm{V} | t', -t' \rangle = \sum_{n \ge 0} \lambda_n(0)\, \pi_n(s', t')$$

For appropriate α the coefficients in the expansion of (I.4) satisfy $\lambda_0(0) < 0$ and $|\lambda_0(0)| \gg |\lambda_n(0)|$, $n \ge 1$. That is, the electron phonon interaction is dominated by the zero angular momentum sector, which is attractive. We assume that this is the case for the rest of this section.

The set of coupling constants $\{\ \lambda_n \ \big|\ n \geq 0\ \}$ evolves under the renormalization group flow. At scale j their values are denoted $\{\ \lambda_n(j) \ \big|\ n \geq 0\ \}$. In the ladder approximation the flow equation is (see [6, I.85])

$$\lambda_n(j-1) \ = \ \lambda_n(j) + \beta(j)\,\lambda_n(j)^2$$

where $\beta(j) > 0$ and $\lim\limits_{j\to-\infty} \beta(j) = \beta > 0$.

Thus, in the ladder approximation, $\lambda_0(j)$ grows slowly as j goes down to the symmetry breaking scale $\delta = -[1/\lambda_0(0)]$ and then quickly takes off to infinity. The other coupling constants remain much smaller than λ_0. Of course, this approximation breaks down at about scale δ. The divergence of a flow generated by a "Fermi surface" away from a Gaussian fixed point towards a nontrivial fixed point is typical of many symmetry breaking or mass generation phenomena in condensed matter physics.

The renormalization group analysis described in the preceding paragraphs reveals three distinct energy regimes. Fix $a \gg 1$ and let $\Delta \approx M^\delta$ be the BCS gap. In the first regime at scales j for which $M^j > a\Delta$ the effective coupling constant $\lambda_0(j)$ can be used as a small parameter. Symmetry breaking takes place in the second regime where $\frac{1}{a}\Delta < M^j < a\Delta$. In the third regime $M^j < \frac{1}{a}\Delta$ the physics of the Goldstone boson dominates. As explained above the effective coupling constant is not small in the latter two regimes.

In [6, 1] the first regime is controlled nonperturbatively in $2+1$ dimensions and perturbatively in $3+1$ dimensions. That is, in $2+1$ dimensions the full model has been constructed down to scale δ and shown to obey the natural estimates suggested by perturbation theory. We remark that it is also shown in [6] that, for a fixed external field that selects a single phase, the renormalization group flow, truncated to any finite order of perturbation theory, converges to a nontrivial pairing fixed point. The main idea for controlling the intermediate regime is an intrinsic decomposition of the Fermi surface into $N = M^{-(d-1)\delta}$ "colors" and the accompanying $1/N$ expansion [3].

This paper is devoted to a tool that is essential in the third regime. Cooper pairs interact with each other through a long range force mediated by the "Goldstone boson". The Goldstone boson is a massless particle, in $d+1$ dimensions, whose propagator behaves like $[q_0^2 + \mathrm{const}\,\mathbf{q}^2]^{-1}$ near zero. This singularity superficially generates nonrenormalizable power counting for $d = 2, 3$. The Ward identities of Corollary IV.5 imply that the power counting is in fact superrenormalizable. A detailed discussion of power counting is given in §V.

The Ward identities of §IV relate certain expectation values. The "string" and "ladder" amputation algorithm developed in §III can be applied to these Ward identities to identify classes of diagrams that add up to zero. This is feasible at low order, but because the symmetry breaking mixes orders, the classes become very complicated. Oppermann and Wegner[7] and Hikami [8] have derived the low order graphical analogues of our Ward identites for the Anderson model. We imagine that the methods of this paper can also be applied to the Anderson model.

The simplest way to see the Goldstone boson is to make a Hubbard-Stratonovich transformation writing the exponential of the effective interaction $\mathcal{V}_{\rm eff}$, given by the renormalization group flow at scale δ, as an integral over an intermediate boson field γ. Let $\lambda_0(\delta) = -2g^2$ and

$$\begin{aligned}\mathcal{V}_{\rm eff} &= -2g^2 \int đs\, đt\, đq\ \bar\psi_{\uparrow(t+\frac{q}{2})}\, \bar\psi_{\downarrow(-t+\frac{q}{2})}\, \psi_{\downarrow(-s+\frac{q}{2})}\, \psi_{\uparrow(s+\frac{q}{2})} \\ &= -2g^2 \int đp\, đq \left(\int đt\, \bar\psi_{\uparrow(t+\frac{p}{2})}\, \bar\psi_{\downarrow(-t+\frac{p}{2})}\right) B(p,-q) \left(\int đs\, \psi_{\downarrow(-s+\frac{q}{2})}\, \psi_{\uparrow(s+\frac{q}{2})}\right)\end{aligned}$$

with $B(p,q) = {\scriptstyle (2\pi)^{d+1}}\delta(p+q)$. If (γ_1, γ_2) is a $\mathbb{C}^2$ valued Gaussian variable with the real, even covariance

$$\langle \gamma_i(p)\gamma_j(q)\rangle = \delta_{i,j}\, B(p,q)$$

then

$$e^{-\mathcal{V}} = e^{-\mathcal{V}+\mathcal{V}_{\rm eff}} e^{-\mathcal{V}_{\rm eff}} = e^{-\mathcal{V}+\mathcal{V}_{\rm eff}} \int \exp\left(g \int d\xi\ \bar{\mathbf{\Psi}}_{(\xi)}\, \gamma_{(\xi)}\, \mathbf{\Psi}_{(\xi)}\right) d\mu_{(\gamma)}$$

where $\gamma = \sigma^1\gamma^1 + \sigma^2\gamma^2$.

The propagator for the Goldstone boson is gotten by integrating out the Fermion field, computing the effective potential for γ and expanding about its minimum. This is done in detail in §V. The effective potential is a Mexican hat. The propagator of the component $\gamma_{\rm tan}$ of γ tangent to the circle of minima looks like $[q_0^2 + {\rm const}\,\mathbf{q}^2]^{-1}$ near zero and is given by

$$\begin{aligned}&\langle \gamma_{\rm tan}(p_1); \gamma_{\rm tan}(p_2)\rangle - (2\pi)^{d+1}\delta(p_1+p_2) \\ &= -g^2 \int đs\, đt\, \langle \bar\psi_{s-p_1\uparrow}\bar\psi_{-s\downarrow} - \psi_{-s+p_1\downarrow}\psi_{s\uparrow}; \bar\psi_{t-p_2\uparrow}\bar\psi_{-t\downarrow} - \psi_{-t+p_2\downarrow}\psi_{t\uparrow}\rangle\end{aligned}$$

Here the truncated expectation $\langle A; B\rangle$ is given by $\langle AB\rangle - \langle A\rangle\langle B\rangle$ and, on the right hand side, $\langle\ \cdot\ \rangle$ is the fermionic expectation of the model before the Hubbard-Stratonovich transformation was applied. This identity, in conjunction with the amputation procedure of §III, is used in (V.4) to demonstrate that the full Goldstone boson propagator is the sum of generalized ladders.

In §II we discuss self-consistent $U(1)$ symmetry breaking and derive a Ward identity using global $U(1)$ symmetry transformations. In §III we determine the structure of Feynman graphs in terms of string and ladder amputation. This makes it possible to isolate the fragments of graphs contributing to the Goldstone Boson propagator. In §IV we derive more precise and general Ward identities exploiting local $U(1)$ gauge transformations. §V is devoted to the power counting of the intermediate boson introduced above. In the last section, we discuss the simple ladder contribution to the Goldstone Boson propagator in great detail.

§II $U(1)$ Symmetry Breaking

We first rewrite the many Fermion generating functional (I.1) in a form that is a suitable starting point for a rigorous construction. Let

$$d\mu(\psi,\bar\psi) \;=\; \tfrac{1}{\mathcal{Z}_0}\exp\Big\{-\sum_{\sigma\in\{\uparrow,\downarrow\}}\int đk\;\bar\psi_{k\sigma}\big(ik_0 e(\mathbf{k})\big)\psi_{k,\sigma}\Big\}\prod_{k,\sigma} d\psi_{k,\sigma}\,d\bar\psi_{k,\sigma}$$

be a formal Grassmann-Gaussian measure. The denominator $\mathcal{Z}_0$ chosen so that

$$\int 1\; d\mu(\psi,\bar\psi) \;=\; 1$$

Mathematically, $d\mu(\psi,\bar\psi)$ is characterized by

$$\int e^{[\bar\phi\psi]+[\bar\psi\phi]}\; d\mu(\psi,\bar\psi) \;=\; e^{<\bar\phi,C\phi>}$$

where, the inner product

$$\langle\bar\phi,C\phi\rangle \;=\; \int d\xi\,d\xi'\;\bar\phi(\xi)\,C(\xi,\xi')\,\phi(\xi')$$

The covariance

$$\begin{aligned}
C(\xi_1,\xi_2) \;&=\; \int \psi(\xi_1)\bar\psi(\xi_2)\,d\mu(\psi,\bar\psi)\\
&=\; \delta_{\sigma_1,\sigma_2}\int_{\mathbb{R}^{d+1}} đk\;\frac{e^{i\langle k,\xi_1-\xi_2\rangle_-}}{ik_0-e(\mathbf{k})}\\
&=\; \delta_{\sigma_1,\sigma_2}\int_{\mathbb{R}^{d}} đ\mathbf{k}\; e^{i\langle \mathbf{k},\mathbf{x}_1-\mathbf{x}_2\rangle} e^{-|e(\mathbf{k})(t_1-t_2)|}\begin{cases}-\theta(e(\mathbf{k})), & t_1>t_2\\ \theta(-e(\mathbf{k})), & t_1\le t_2\end{cases}
\end{aligned}$$

where

$$\theta(s) \;=\; \begin{cases}1, & s>0\\ 0, & s\le 0\end{cases}$$

We have

$$\mathcal{S}(\phi,\bar\phi) \;=\; \log\tfrac{1}{\mathcal{Z}}\int e^{[\bar\phi\psi]+[\bar\psi\phi]}\; e^{-\mathcal{V}(\psi,\bar\psi)\;-\;\delta\mu(\lambda,\mu)\int đk\,\bar\psi_k\psi_k}\; d\mu(\psi,\bar\psi) \qquad \text{(II.1)}$$

The proper self-energy, Σ, is implicitly defined by

$$S_2(k,p) \;=\; \frac{1}{ik_0-e(\mathbf{k})-\Sigma(k)}\;(2\pi)^{d+1}\delta(k-p)$$

in which

$$S_2(\xi_1,\sigma_1,\xi_2,\sigma_2) \;=\; \frac{\delta}{\delta\bar\phi(\xi_1,\sigma_1)}\,\mathcal{S}(\phi,\bar\phi)\,\frac{\delta}{\delta\phi(\xi_2,\sigma_2)}\Big|_{\phi=\bar\phi=0}$$

The functional derivative $\frac{\delta}{\delta\bar\phi(\xi_1,\sigma_1)}$ acting on the left of a monomial moves the factor $\bar\phi(\xi_1,\sigma_1)$ all the way to the left with the appropriate sign and deletes it. Similarly, the functional derivative $\frac{\delta}{\delta\phi(\xi_2,\sigma_2)}$ acting on the right of a monomial moves the factor $\phi(\xi_2,\sigma_2)$ all the way to the right with the appropriate sign and deletes it. For example,

$$\begin{aligned}\frac{\delta}{\delta\bar\phi(\xi_1,\sigma_1)}\, e^{[\bar\phi\psi]+[\bar\psi\phi]}\, \frac{\delta}{\delta\phi(\xi_2,\sigma_2)} &= \psi(\xi_1,\sigma_1)\, e^{[\bar\phi\psi]+[\bar\psi\phi]}\, \frac{\delta}{\delta\phi(\xi_2,\sigma_2)}\\ &= \psi(\xi_1,\sigma_1)\, e^{[\bar\phi\psi]+[\bar\psi\phi]}\, \bar\psi(\xi_2,\sigma_2)\\ &= \psi(\xi_1,\sigma_1)\bar\psi(\xi_2,\sigma_2)\, e^{[\bar\phi\psi]+[\bar\psi\phi]}\end{aligned}$$

so that

$$\frac{\delta}{\delta\bar\phi(\xi_1,\sigma_1)}\, \mathcal{S}(\phi,\bar\phi)\, \frac{\delta}{\delta\phi(\xi_2,\sigma_2)}\Big|_{\phi=\bar\phi=0} = \big\langle \psi(\xi_1,\sigma_1)\bar\psi(\xi_2,\sigma_2)\big\rangle - \langle\psi(\xi_1,\sigma_1)\rangle\,\big\langle\bar\psi(\xi_2,\sigma_2)\big\rangle$$

where

$$\big\langle f(\psi,\bar\psi)\big\rangle = \tfrac{1}{\mathcal{Z}}\int f(\psi,\bar\psi)\; e^{-\mathcal{V}(\psi,\bar\psi)\ -\ \delta\mu(\lambda,\mu)\int đk\,\bar\psi_k\psi_k}\, d\mu(\psi,\bar\psi)$$

Perturbatively, Σ is the sum of all nontrivial one-particle irreducible, two legged diagrams.

The counterterm $\delta\mu(\lambda,\mu)$ is the formal power series in λ uniquely determined [5,6] by the renormalization condition

$$\frac{\partial^n}{\partial\lambda^n}\,\Sigma(0,|\mathbf{k}|=k_F,\mu,\lambda)\Big|_{\lambda=0} = 0\ ,\quad n\ge 0$$

where $k_F=\sqrt{2\mathbf{m}\mu}$ is fixed by the particle density. In other words, the proper self-energy vanishes on $k_0=0$, $|\mathbf{k}|=k_F$ and, by definition, k_F is the radius of the interacting Fermi surface.

Observe that

$$e^{-\delta\mu(\lambda,\mu)\int đp\,\bar\psi_p\psi_p}\,d\mu(\psi,\bar\psi) = \tfrac{1}{\mathcal{Z}_0}\, e^{-\int đk\ \left(ik_0 e(\mathbf{k},\mu+\delta\mu)\right)\bar\psi_k\psi_k}\prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma}$$

Therefore, the counterterm can be interpreted as the shift in the radius of the Fermi surface induced by $\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle$.

We now state one of our goals precisely. To keep the statement simple, we use periodic boundary conditions on $\mathbb{R}^{d+1}/L\mathbb{Z}^{d+1}$. Define $d\mu_L$ to be the Grassmann Gaussian measure whose covariance is the multiplication operator

$$\frac{1}{ik_0-e(\mathbf{k})}\big(1-\delta_{k_0,0}\delta_{e(\mathbf{k}),0}\big)$$

on $\ell^2\big(\frac{2\pi}{L}\mathbb{Z}^{d+1}\big)$. For convenience let

$$\int đk\ f(k) = \big(\tfrac{2\pi}{L}\big)^{d+1}\sum_{k\in\frac{2\pi}{L}\mathbb{Z}^{d+1}} f(k)$$

Incorporate in the finite volume interaction $\mathcal{V}_{L,r}$ a counterterm for the chemical potential and a small external field

$$\mathcal{V}_{L,r}(\psi,\bar\psi) = \mathcal{V}(\psi,\bar\psi) + \delta\mu(\lambda,\mu;L,r)\int \bar\psi_k\psi_k - r\int đk(\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow} + \psi_{-k\downarrow}\psi_{k\uparrow})$$

and define the generating functional

$$\mathcal{S}_{L,r}(\phi,\bar\phi) \;=\; \log \tfrac{1}{\mathcal{Z}_{L,r}} \int e^{[\bar\phi\psi]+[\bar\psi\phi]}\, e^{-\mathcal{V}_{L,r}(\psi,\bar\psi)}\, d\mu_L(\psi,\bar\psi)$$

We ultimately want to prove the

"Theorem". *Let $d = 2,3$ and let $\langle k_1,k_2|V|k_3,k_4\rangle$ be a Schwartz class function on $\mathbb{R}^{4(d+1)}$ satisfying symmetries (S1,2) of §I. (Actually, it is sufficient that the interaction be in a suitable weighted Sobolev space.) Let*

$$\langle t',-t'|V|s',-s'\rangle = \sum_n \lambda_n\pi_n(t',s')$$

be the expansion of the rotation invariant reduced kernel in spherical harmonics. Here $k' = (0, k_F\mathbf{k}/|\mathbf{k}|)$ is the projection of k on the Fermi surface. Fix $\epsilon > 0$. Let $\lambda > 0$ and κ be sufficiently small. If $\lambda_0 < 0$ and $\kappa|\lambda_0| > |\lambda_n|,\ n \ge 1$ then the limit

$$\mathcal{S}(\phi,\bar\phi) = \lim_{r\searrow 0}\lim_{L\to\infty} \mathcal{S}_{L,r}(\phi,\bar\phi)$$

exists and has the following properties:

(i) There is a $\Delta > 0$ with $\Delta \approx \mathrm{const}\, e^{-\mathrm{const}/\lambda}$ such that

$$\langle\psi_{k'\uparrow}\psi_{-p\downarrow}\rangle = \langle\bar\psi_{-k'\downarrow}\bar\psi_{p\uparrow}\rangle = -\frac{(2\pi)^{d+1}}{\Delta}\delta(k'-p)$$

(ii) The $2n$ point moments of $S(\phi,\bar\phi)$ with n odd decay exponentially at a rate at least $(1-\epsilon)\Delta$.

(iii) The $2n$ point moments of $S(\phi,\bar\phi)$ with n even decay at least polynomially. In particular, there are constants $c_1, c_2 > 0$ such that

$$\lim_{q\to 0}(c_1q_0^2 + c_2\mathbf{q}^2)\int đs\, đt\, đp\,\langle\bar\psi_{s-q\uparrow}\bar\psi_{-s\downarrow} - \psi_{-s+q\downarrow}\psi_{s\uparrow};\bar\psi_{t-p\uparrow}\bar\psi_{-t\downarrow} - \psi_{-t+p\downarrow}\psi_{t\uparrow}\rangle = -1$$

Thus there is a channel in the four point function that does not decay exponentially.

One consequence of this Theorem is that the Hilbert space is a direct sum of "even" and "odd" subspaces. The restriction of the Hamiltonian to the odd subspace has a gap of at least $(1-\epsilon)\Delta$ between its ground state energy and the rest of its spectrum. There is no such gap in the even subspace.

The rest of this section motivates the "Theorem". The action

$$\begin{aligned}\mathcal{A}(\psi,\bar\psi) \;=\; & -\int đk\ \big(ik_0 e(\mathbf{k})\big)\bar\psi_k\psi_k \;-\; \delta\mu(\lambda,\mu)\int đk\ \bar\psi_k\psi_k \\ & -\tfrac{\lambda}{2}\int\prod_{i=1}^{4} đk_i\ (2\pi)^{d+1}\delta(k_1{+}k_2{-}k_3{-}k_4)\ \bar\psi_{k_1}\psi_{k_3}\,\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle\,\bar\psi_{k_2}\psi_{k_4}\end{aligned}$$

has six basic symmetries. Namely
(i) Particle number:

$$\mathcal{A}(e^{i\theta}\psi, e^{-i\theta}\bar\psi) = \mathcal{A}(\psi,\bar\psi) \qquad \forall\ e^{i\theta}\in U(1)$$

(ii) Spin:

$$\mathcal{A}(g\,\psi,\bar\psi\,g^{-1}) = \mathcal{A}(\psi,\bar\psi) \qquad \forall\ g\in SU(2)$$

(iii) Spatial rotations and reflections:

$$\mathcal{A}(R\psi,R\bar\psi) = \mathcal{A}(\psi,\bar\psi) \qquad \forall\ R\in O(d)$$

where $(R\psi)(\xi,\sigma) = \psi(R^{-1}\xi,\sigma)$ and $(R\bar\psi)(\xi,\sigma) = \bar\psi(R^{-1}\xi,\sigma)$.
(iv) Translations: For all $\xi\in\mathbb{R}^{d+1}$,

$$\mathcal{A}(T_\xi\psi,T_\xi\bar\psi) = \mathcal{A}(\psi,\bar\psi) \qquad \forall\ \xi\in\mathbb{R}^{d+1}$$

where $(T_\xi\psi)_{k,\sigma} = e^{i<k,\xi>_-}\psi_{k,\sigma}$ and $(T_\xi\bar\psi)_{k,\sigma} = e^{-i<k,\xi>_-}\bar\psi_{k,\sigma}$
(v) Time reversal:

$$\mathcal{A}(\psi,\bar\psi)^{\#} = \mathcal{A}(\psi,\bar\psi)$$

where # is the involution on the Grassmann algebra defined by $\psi_k^{\#} = \psi_{Tk}$, $\bar\psi_k^{\#} = \bar\psi_{Tk}$ and by complex conjugation of scalars.
(vi) Charge conjugation:

$$\mathcal{A}(i\bar\psi^t, i\psi^t) = \mathcal{A}(\psi,\bar\psi)$$

To verify (v), note that

$$\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle \;=\; \langle k_3,k_4|\mathrm{V}|k_1,k_2\rangle$$

follows from the reflection invariance and symmetry (S2) of V. Observe that, in contrast to the other symmetries, neither time reversal nor charge conjugation commute with the number symmetry. However, their product
(vii) CT:

$$\mathcal{A}(\psi,\bar\psi)^{\mathrm{CT}} = \mathcal{A}(\psi,\bar\psi)$$

does commute with the number symmetry. Here, CT is the involution on the Grassmann algebra defined by $\psi_k^{\mathrm{CT}} = i\bar\psi_{Tk}^t$, $\bar\psi_k^{\mathrm{CT}} = i\psi_{Tk}^t$ and by complex conjugation of scalars.

By definition, a general symmetry $\mathcal{U}$ of the action is broken if

$$\mathcal{A}\big(\mathcal{U}(\psi,\bar\psi)\big) \;=\; \mathcal{A}(\psi,\bar\psi) \qquad \text{but} \qquad \mathcal{S}\big(\mathcal{U}(\phi,\bar\phi)\big) \;\neq\; \mathcal{S}(\phi,\bar\phi)$$

where $\mathcal{S}$ is the generating functional (I.1), carefully defined by some limiting process. In this paper we study the situation when the number symmetry is broken

$$\mathcal{S}(e^{i\theta}\phi, e^{-i\theta}\bar\phi) \;\neq\; \mathcal{S}(\phi,\bar\phi)$$

but symmetries (ii,iii,iv) and (vii) above are inherited by the generating functional.

The symmetry

(viii):

$$\begin{pmatrix}\psi_{k,\uparrow}\\ \psi_{k,\downarrow}\end{pmatrix} \mapsto \begin{pmatrix}e^{i\phi} & 0\\ 0 & e^{-i\phi}\end{pmatrix}\begin{pmatrix}\psi_{k,\uparrow}\\ \psi_{k,\downarrow}\end{pmatrix}$$
$$(\,\bar\psi_{k,\uparrow} \quad \bar\psi_{k,\downarrow}\,) \mapsto (\,\bar\psi_{k,\uparrow} \quad \bar\psi_{k,\downarrow}\,)\begin{pmatrix}e^{-i\phi} & 0\\ 0 & e^{i\phi}\end{pmatrix}$$

is a subgroup of $SU(2)$. Applying this symmetry to the first derivatives of $\mathcal{S}$ we obtain

$$\left\langle \overset{(-)}{\psi}_{k,\sigma}\right\rangle \;=\; e^{\pm i\phi}\left\langle \overset{(-)}{\psi}_{k,\sigma}\right\rangle$$

forcing

$$\langle\psi_{k,\sigma}\rangle \;=\; \langle\bar\psi_{k,\sigma}\rangle = 0$$

for all k, σ. Thus the second derivative of $\mathcal{S}$ simplifies to, for example,

$$\frac{\delta}{\delta\bar\phi(\xi_1,\sigma_1)}\,\mathcal{S}(\phi,\bar\phi)\,\frac{\delta}{\delta\phi(\xi_2,\sigma_2)}\Big|_{\phi=\bar\phi=0} \;=\; \big\langle\psi(\xi_1,\sigma_1)\bar\psi(\xi_2,\sigma_2)\big\rangle$$

There are sixteen two point expectation values

$$\begin{array}{cccc}
\langle\psi_{k\uparrow}\psi_{p\uparrow}\rangle & \langle\psi_{k\uparrow}\psi_{p\downarrow}\rangle & \langle\psi_{k\downarrow}\psi_{p\uparrow}\rangle & \langle\psi_{k\downarrow}\psi_{p\downarrow}\rangle\\
\langle\psi_{k\uparrow}\bar\psi_{p\uparrow}\rangle & \langle\psi_{k\uparrow}\bar\psi_{p\downarrow}\rangle & \langle\psi_{k\downarrow}\bar\psi_{p\uparrow}\rangle & \langle\psi_{k\downarrow}\bar\psi_{p\downarrow}\rangle\\
\langle\bar\psi_{k\uparrow}\psi_{p\uparrow}\rangle & \langle\bar\psi_{k\uparrow}\psi_{p\downarrow}\rangle & \langle\bar\psi_{k\downarrow}\psi_{p\uparrow}\rangle & \langle\bar\psi_{k\downarrow}\psi_{p\downarrow}\rangle\\
\langle\bar\psi_{k\uparrow}\bar\psi_{p\uparrow}\rangle & \langle\bar\psi_{k\uparrow}\bar\psi_{p\downarrow}\rangle & \langle\bar\psi_{k\downarrow}\bar\psi_{p\uparrow}\rangle & \langle\bar\psi_{k\downarrow}\bar\psi_{p\downarrow}\rangle
\end{array}$$

obtained by differentiating $\mathcal{S}$ twice with respect to $\overset{(-)}{\phi}(\xi,\sigma)$. By conservation of momentum, that is translation invariance, the distributions in the first and fourth rows vanish unless $k=-p$ while those in the second and third vanish unless $k=p$. By anticommutation, the second row determines the third row and the second column determines the third column.

The four corners, $\langle\psi_{k\uparrow}\psi_{p\uparrow}\rangle\,,\langle\psi_{k\downarrow}\psi_{p\downarrow}\rangle\,,\langle\bar\psi_{k\uparrow}\bar\psi_{p\uparrow}\rangle\,\langle\bar\psi_{k\downarrow}\bar\psi_{p\downarrow}\rangle$, of the above table and the four central elements, $\langle\psi_{k\uparrow}\bar\psi_{p\downarrow}\rangle\,,\langle\psi_{k\downarrow}\bar\psi_{p\uparrow}\rangle\,,\langle\bar\psi_{k\uparrow}\psi_{p\downarrow}\rangle\,,\langle\bar\psi_{k\downarrow}\psi_{p\uparrow}\rangle$, vanish by $SU(2)$ invariance. For example, by symmetry (viii)

$$\langle\psi_{k\uparrow}\bar\psi_{p\downarrow}\rangle \;=\; e^{2i\alpha}\,\langle\psi_{k\uparrow}\bar\psi_{p\downarrow}\rangle$$

forcing it to be zero. The other cases are similar. Thus there are eight

$$\begin{pmatrix} \langle\psi_{k\uparrow}\bar\psi_{p\uparrow}\rangle & \langle\psi_{k\uparrow}\psi_{-p\downarrow}\rangle \\ \langle\bar\psi_{-k\downarrow}\bar\psi_{p\uparrow}\rangle & \langle\bar\psi_{-k\downarrow}\psi_{-p\downarrow}\rangle \end{pmatrix} \qquad\qquad \begin{pmatrix} \langle\bar\psi_{k\uparrow}\psi_{p\uparrow}\rangle & \langle\psi_{-k\downarrow}\psi_{p\uparrow}\rangle \\ \langle\bar\psi_{k\uparrow}\bar\psi_{-p\downarrow}\rangle & \langle\psi_{-k\downarrow}\bar\psi_{-p\downarrow}\rangle \end{pmatrix}$$

potentially nonzero two point expectation values with the second matrix determined by the first and all matrix elements vanishing unless $k = p$.

For this reason it is algebraically convenient to combine the four internal physical fields $\psi_{k\uparrow}$, $\psi_{-k\downarrow}$, $\bar\psi_{k\uparrow}$ and $\bar\psi_{-k\downarrow}$ into a pair of 2-vectors

$$\begin{aligned} \mathbf{\Psi}(k) &= \begin{pmatrix} \mathbf{\Psi}^1(k) \\ \mathbf{\Psi}^2(k) \end{pmatrix} = \begin{pmatrix} \psi_{k\uparrow} \\ \bar\psi_{-k\downarrow} \end{pmatrix} \\ \bar{\mathbf{\Psi}}(k) &= (\, \bar{\mathbf{\Psi}}_1(k) \quad \bar{\mathbf{\Psi}}_2(k)\,) = (\, \bar\psi_{k\uparrow} \quad \psi_{-k\downarrow}\,) \end{aligned}$$

called "Nambu fields". The external physical fields are combined into

$$\begin{aligned} \mathbf{\Phi}(k) &= \begin{pmatrix} \mathbf{\Phi}^1(k) \\ \mathbf{\Phi}^2(k) \end{pmatrix} = \sigma^3 \begin{pmatrix} \phi_{k\uparrow} \\ \bar\phi_{-k\downarrow} \end{pmatrix} \\ \bar{\mathbf{\Phi}}(k) &= (\, \bar{\mathbf{\Phi}}_1(k) \quad \bar{\mathbf{\Phi}}_2(k)\,) = (\, \bar\phi_{k\uparrow} \quad \phi_{-k\downarrow}\,)\,\sigma^3 \end{aligned}$$

Note that the external fields are twisted by σ^3, the third of the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

With this vector notation all potentially nonzero expectation values are contained in

$$\Big(\big\langle \mathbf{\Psi}^i(k)\bar{\mathbf{\Psi}}_j(p)\big\rangle \Big) = \begin{pmatrix} \langle\psi_{k\uparrow}\bar\psi_{p\uparrow}\rangle & \langle\psi_{k\uparrow}\psi_{-p\downarrow}\rangle \\ \langle\bar\psi_{-k\downarrow}\bar\psi_{p\uparrow}\rangle & \langle\bar\psi_{-k\downarrow}\psi_{-p\downarrow}\rangle \end{pmatrix} \;=\; (2\pi)^{d+1}\delta(k-p)S(k)$$

and

$$\Big(\big\langle \bar{\mathbf{\Psi}}_i(k)\mathbf{\Psi}^j(p)\big\rangle \Big) \;=\; -(2\pi)^{d+1}\delta(k-p)S(k)^t$$

while

$$\Big(\big\langle \mathbf{\Psi}^i(k)\mathbf{\Psi}^j(p)\big\rangle \Big) = \begin{pmatrix} \langle\psi_{k\uparrow}\psi_{p\uparrow}\rangle & \langle\psi_{k\uparrow}\bar\psi_{-p\downarrow}\rangle \\ \langle\bar\psi_{-k\downarrow}\psi_{p\uparrow}\rangle & \langle\bar\psi_{-k\downarrow}\bar\psi_{-p\downarrow}\rangle \end{pmatrix}$$

and $\Big(\big\langle \bar{\mathbf{\Psi}}_i(k)\bar{\mathbf{\Psi}}_j(p)\big\rangle \Big)$ remain identically zero.

Lemma II.1 *Suppose the generating functional $\mathcal{S}$ inherits the symmetries (ii,iii,iv) and (vii) from the action $\mathcal{A}$. Then*

$$\Big(\big\langle \mathbf{\Psi}^i(k)\mathbf{\Psi}^j(p)\big\rangle \Big) \;=\; \Big(\big\langle \bar{\mathbf{\Psi}}_i(k)\bar{\mathbf{\Psi}}_j(p)\big\rangle \Big) \;=\; 0$$

Furthermore

$$\begin{aligned} S(k_0,\mathbf{k}) &= S(k_0,|\mathbf{k}|) & \qquad S(0,\mathbf{k}) &= \overline{S(0,\mathbf{k})}^* \\ S_{22}(k) &= -S_{11}(Tk) & S_{11}(k) &= \overline{S_{11}}(Tk) \\ S_{22}(k) &= \overline{S_{22}}(Tk) & S_{12}(k) &= \overline{S_{21}}(Tk) \end{aligned}$$

Proof: The first statement has already been proven. The identity

$$S(k_0, \mathbf{k}) = S(k_0, |\mathbf{k}|)$$

is an immediate consequence of the symmetry (iii). Next

$$\begin{aligned} S_{22}(k)(2\pi)^{d+1}\delta(k-p) &= \left\langle \bar\psi_{-k\downarrow}\psi_{-p\downarrow}\right\rangle = \left\langle \bar\psi_{-k\uparrow}\psi_{-p\uparrow}\right\rangle = -\left\langle \psi_{-p\uparrow}\bar\psi_{-k\uparrow}\right\rangle \\ &= -\left\langle \psi_{Tp\uparrow}\bar\psi_{Tk\uparrow}\right\rangle = -S_{11}(Tp)(2\pi)^{d+1}\delta(Tp-Tk) \\ &= -S_{11}(Tk)(2\pi)^{d+1}\delta(k-p) \end{aligned}$$

In passing from the first to the second line we used $SU(2)$ invariance with

$$g = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

To pass from the third to the fourth line we used invariance under spatial reflection.

Similarly,

$$\begin{aligned} S_{12}(k)(2\pi)^{d+1}\delta(k-p) &= \langle \psi_{k\uparrow}\psi_{-p\downarrow}\rangle = -\overline{\left\langle \bar\psi_{Tk\uparrow}\bar\psi_{-Tp\uparrow}\right\rangle} = \overline{\left\langle \bar\psi_{-Tp\uparrow}\bar\psi_{Tk\uparrow}\right\rangle} \\ &= \overline{S_{21}(Tp)}(2\pi)^{d+1}\delta(Tp-Tk) \\ &= \overline{S_{21}(Tk)}(2\pi)^{d+1}\delta(k-p) \end{aligned}$$

We used CT invariance in the second line. The proof of the third and fourth identities is similar.

The third fourth and fifth identities may be combined to give $S(k) = S(Tk)^*$. The last identity follows at once. ■

Formally, the Grassmann-Gaussian measure in Nambu fields is

$$d\mu_{(\mathbf{\Psi},\bar{\mathbf{\Psi}})} = \tfrac{1}{\mathcal{Z}_0}\exp\left\{-\int đk\ \bar{\mathbf{\Psi}}(k)\big(ik_0\mathbb{1}e(\mathbf{k})\sigma^3\big)\mathbf{\Psi}(k)\right\}\prod_{k,i} d\mathbf{\Psi}^i(k)\, d\bar{\mathbf{\Psi}}_i(k)$$

and is rigorously characterized by its characteristic functional

$$\int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}]+[\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\, d\mu_{(\mathbf{\Psi},\bar{\mathbf{\Psi}})} = e^{<\bar{\mathbf{\Phi}},\mathbf{C}\mathbf{\Phi}>}$$

where the covariance

$$\Big(\left\langle \mathbf{\Psi}^i(k)\bar{\mathbf{\Psi}}_j(p)\right\rangle\Big) = (2\pi)^{d+1}\delta(k-p)C(k)$$

with

$$C(k) = \frac{1}{ik_0\mathbb{1}-e(\mathbf{k})\sigma^3} = -\frac{ik_0\mathbb{1}+e(\mathbf{k})\sigma^3}{k_0^2+e(\mathbf{k})^2} = \begin{pmatrix} [ik_0-e(\mathbf{k})]^{-1} & 0 \\ 0 & [ik_0+e(\mathbf{k})]^{-1} \end{pmatrix}$$

The generating functional (I.1) becomes, in Nambu fields,

$$\mathcal{S}(\mathbf{\Phi},\bar{\mathbf{\Phi}}) = \log \tfrac{1}{\mathcal{Z}} \int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\ e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\delta\mu(\lambda,\mu)\int đk\,\bar{\mathbf{\Psi}}(k)\sigma^3\mathbf{\Psi}(k)}\, d\mu(\mathbf{\Psi},\bar{\mathbf{\Psi}}) \tag{II.2}$$

where

$$\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}}) \ = \tfrac{\lambda}{2}\int đs\, đt\, đq\, \left(\bar{\mathbf{\Psi}}(t+\tfrac{q}{2})\sigma^3\mathbf{\Psi}(s+\tfrac{q}{2})\right) \left\langle t+\tfrac{q}{2},-t+\tfrac{q}{2}\middle| V \middle| s+\tfrac{q}{2},-s+\tfrac{q}{2}\right\rangle \left(\bar{\mathbf{\Psi}}(-t+\tfrac{q}{2})\sigma^3\mathbf{\Psi}(-s+\tfrac{q}{2})\right) \tag{II.3}$$

Define the real numbers Δ_1 and Δ_2 by

$$\frac{\Delta_1 - i\Delta_2}{\Delta_1^2+\Delta_2^2} \ = \ -S_{12}(0,|\mathbf{k}|=k_F)$$

By Lemma II.1

$$\frac{\Delta_1 + i\Delta_2}{\Delta_1^2+\Delta_2^2} \ = \ -S_{21}(0,|\mathbf{k}|=k_F)$$

Let $d\mu_\Delta$ be the Grassmann-Gaussian measure with covariance

$$\mathbf{C}_\Delta \ = \ \frac{1}{ik_0\mathbb{1} - e(\mathbf{k})\sigma^3 - \Delta} \ = \ -\frac{ik_0\mathbb{1} + e(\mathbf{k})\sigma^3 + \Delta}{k_0^2+E(\mathbf{k})^2}$$

where

$$\Delta \ = \ \Delta_1\sigma^1 + \Delta_2\sigma^2$$

and

$$E(\mathbf{k})^2 \ = \ e(\mathbf{k})^2+\Delta^2 \ = \ e(\mathbf{k})^2+\Delta_1^2+\Delta_2^2$$

When $k_0=0$ and $|\mathbf{k}|=k_F$ the off-diagonal components of

$$\int \mathbf{\Psi}(k)\bar{\mathbf{\Psi}}(p)\ d\mu_\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}}) \ = \ -\frac{1}{k_0^2+E(\mathbf{k})^2}\begin{pmatrix} ik_0+e(\mathbf{k}) & \Delta_1-i\Delta_2 \\ \Delta_1+i\Delta_2 & ik_0-e(\mathbf{k})\end{pmatrix}(2\pi)^{d+1}\delta(k-p)$$

are exactly the off diagonal components of $S(k)$.

We want to treat the interacting Fermionic measure as a perturbation of $d\mu_\Delta$. For this reason, we multiply and divide by

$$e^{\int đk\ \bar{\mathbf{\Psi}}(k)\Delta\mathbf{\Psi}(k)}$$

to obtain

$$\begin{aligned}\mathcal{S}(\mathbf{\Phi},\bar{\mathbf{\Phi}}) \ &= \ \log\tfrac{1}{\mathcal{Z}}\int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\ e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\delta\mu[\bar{\mathbf{\Psi}}\sigma^3\mathbf{\Psi}]}\,d\mu \\ &= \ \log\tfrac{1}{\mathcal{Z}}\int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\ e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\delta\mu[\bar{\mathbf{\Psi}}\sigma^3\mathbf{\Psi}]-[\bar{\mathbf{\Psi}}\Delta\mathbf{\Psi}]}e^{[\bar{\mathbf{\Psi}}\Delta\mathbf{\Psi}]}\,d\mu \\ &= \ \log\tfrac{1}{\mathcal{Z}_\Delta}\int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\ e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\delta\mu[\bar{\mathbf{\Psi}}\sigma^3\mathbf{\Psi}]-[\bar{\mathbf{\Psi}}\Delta\mathbf{\Psi}]}\,d\mu_\Delta\end{aligned}$$

Define a new proper self-energy by

$$\mathbf{S}_2(k,p) = \frac{1}{ik_0 - e(\mathbf{k})\sigma^3 - \Delta - \mathbf{\Sigma}(k)}\ (2\pi)^{d+1}\delta(k-p)$$

Inverting,

$$\mathbf{\Sigma}(k) \;=\; S^{-1}(k) - ik_0 + e(\mathbf{k})\sigma^3 + \Delta$$

To be consistent with the definition of the physical chemical potential μ it is necessary that

$$\mathbf{\Sigma}_{11}(0,k_F) = \mathbf{\Sigma}_{22}(0,k_F) = 0$$

To be consistent with the definitions of Δ_1 and Δ_2 it is necessary that

$$\begin{aligned} S_{12}(0,k_F) &= \mathbf{C}_{12}(0,k_F) \\ S_{21}(0,k_F) &= \mathbf{C}_{21}(0,k_F) \end{aligned}$$

Since $\mathbf{C}_{11}(0,k_F) = \mathbf{C}_{22}(0,k_F) = 0$ and

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}^{-1} = \frac{1}{\det S}\begin{pmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{pmatrix}$$

the conditions on $\mathbf{\Sigma}_{11}(0,k_F)$, $\mathbf{\Sigma}_{22}(0,k_F)$, $S_{12}(0,k_F)$ and $S_{21}(0,k_F)$ may be combined in

$$\mathbf{\Sigma}(0,k_F) \;=\; 0 \tag{II.4}$$

We must renormalize to ensure that the condition (II.4) is fulfilled. That is, we introduce the renormalized action

$$\mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}}) \;=\; -\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}}) - \int đk\ \bar{\mathbf{\Psi}}_k D(\lambda,\mu,\Delta)\mathbf{\Psi}_k - \int đk\ \bar{\mathbf{\Psi}}_k\big(ik_0 e(\mathbf{k})\sigma^3 - \Delta\big)\mathbf{\Psi}_k$$

and generating functional

$$\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}}) \;=\; \log \tfrac{1}{\mathcal{Z}_\Delta} \int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\ e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\int\, đk\ \bar{\mathbf{\Psi}}(k)D\mathbf{\Psi}(k)}\, d\mu_\Delta \tag{II.5}$$

One can prove [6] that, for each Δ_1, Δ_2 and μ, the counterterm

$$D = D_1(\lambda,\mu,\Delta)\sigma^1 + D_2(\lambda,\mu,\Delta)\sigma^2 + D_3(\lambda,\mu,\Delta)\sigma^3$$

is uniquely determined as a formal power series in λ by the renormalization condition (II.4). The coefficient D_3 is the difference between the bare and physical chemical potentials. On the other hand, there are no physical parameters to shift to accomodate D_1 and D_2. Therefore, the constraints

$$\begin{aligned} D_1(\lambda,\mu,\Delta) &= \Delta_1 \\ D_2(\lambda,\mu,\Delta) &= \Delta_2 \end{aligned} \tag{II.6}$$

must be imposed to ensure that

$$\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}})=\mathcal{S}(\mathbf{\Phi},\bar{\mathbf{\Phi}}) \tag{II.7}$$

The constaints (II.6) are a nonperturbative analogue of the BCS gap equation.

In the Nambu notation, the number symmetry is given by

$$\begin{aligned}\mathbf{\Psi}(k)&\mapsto e^{i\theta\sigma^3}\mathbf{\Psi}(k)\\ \bar{\mathbf{\Psi}}(k)&\mapsto \bar{\mathbf{\Psi}}(k)e^{-i\theta\sigma^3}\end{aligned}$$

The result of applying the number symmetry to a quadratic monomial is

$$\begin{aligned}\bar{\mathbf{\Psi}}(k)\sigma^j\mathbf{\Psi}(p)&\mapsto \bar{\mathbf{\Psi}}(k)e^{-i\theta\sigma^3}\sigma^j e^{i\theta\sigma^3}\mathbf{\Psi}(p)\\ &=\bar{\mathbf{\Psi}}(k)\left\{\begin{array}{ll}\sigma^1\cos 2\theta+\sigma^2\sin 2\theta & j=1\\ -\sigma^1\sin 2\theta+\sigma^2\cos 2\theta & j=2\\ \sigma^3 & j=3\end{array}\right\}\mathbf{\Psi}(p)\end{aligned}$$

so that

$$\begin{aligned}\mathcal{A}_R(e^{i\theta\sigma^3}\mathbf{\Psi},\bar{\mathbf{\Psi}}e^{-i\theta\sigma^3}) &= -\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\int đk\ \bar{\mathbf{\Psi}}_k R(2\theta)D(\lambda,\mu,\Delta)\mathbf{\Psi}_k\\ &\quad -\int đk\ \bar{\mathbf{\Psi}}_k\big(ik_0 e(\mathbf{k})\sigma^3-R(2\theta)\Delta\big)\mathbf{\Psi}_k\end{aligned}$$

where

$$R(\theta)(\sigma\cdot v)=\sigma\cdot\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}$$

Recalling that the counterterms are uniquely determined as formal power series, we obtain

$$D(\lambda,\mu,R(2\theta)\Delta)=R(2\theta)D(\lambda,\mu,\Delta)$$

It follows from the last identity that the set of solutions of (II.6) is invariant under rotations. If (II.6) has a nonzero solution then

$$D_1^2(\lambda,\mu,\Delta)+D_2^2(\lambda,\mu,\Delta)=\Delta_1^2+\Delta_2^2$$

determines $\Delta^2=\Delta_1^2+\Delta_2^2$ as a function of λ and μ, but Δ_1/Δ_2 is completely free.

We have shown that the assumption that

$$S_2(0,k_F)=\mathbf{C}_\Delta(0,k_F)$$

forces $|\Delta|$ to be a solution of the BCS gap equation. However, the phase of Δ has not been determined. It must be fixed externally by imposing, for example, boundary conditions. To select a particular solution Δ of (II.6) we introduce a term $r\Delta,\ r>0$ in (II.2)

$$\begin{aligned}\mathcal{S}(\mathbf{\Phi},\bar{\mathbf{\Phi}},r)&=\log\tfrac{1}{\mathcal{Z}}\int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\ e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})+\int đk\ \bar{\mathbf{\Psi}}(k)\,[r\Delta-\delta\mu\sigma^3]\mathbf{\Psi}(k)}\,d\mu(\mathbf{\Psi},\bar{\mathbf{\Psi}})\\ &=\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}},r)\end{aligned}$$

where

$$\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}},r) \;=\; \log \tfrac{1}{\mathcal{Z}_\Delta} \int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\; e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\int đk\;\bar{\mathbf{\Psi}}(k)\,(D-r\Delta)\mathbf{\Psi}(k)}\, d\mu_{\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}})}$$

provided

$$D_1(\lambda,\mu,\Delta,r)=\Delta_1$$
$$D_2(\lambda,\mu,\Delta,r)=\Delta_2$$

The counterterm D is now determined by the r dependent renormalization condition

$$\Sigma(0,k_F,r)=0$$

The limit,

$$\lim_{r\to 0+} \mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}},r) \tag{II.8}$$

subject to the constraint, imposes "boundary conditions" on (II.2).

To give mathematical meaning to (II.8) we regularize and take an additional limit. For convenience we use a slice cutoff on a continuous momentum space rather than the periodic boundary conditions in the statemnet of the "Theorem". Precisely, fix $M \gg 1$ and introduce, for each integer $h \le 0$, the regularized propagator

$$\rho_h(k)\,\mathbf{C}_\Delta(k) \;=\; \frac{\rho_h(k)}{ik_0 - e(\mathbf{k})\sigma^3 - \Delta} \tag{II.9}$$

where

$$\rho_h(k) \;=\; \sum_{0\ge i\ge h} f\big(M^{-2i}(k_0^2+E(\mathbf{k})^2)\big)$$

and $f(x)$, $x\ge 0$, is a smooth function satisfying

$$f(x) \;=\; \begin{cases} 0 & x\le 1 \text{ or } x\ge \frac{1}{2}M^4 \\ 1 & \frac{1}{2}M^2 \le x \le M^2 \end{cases}$$

$$1 \;=\; \sum_{i\in\mathbb{Z}} f(M^{-2i}x)$$

Observe that $\rho_h(k)$ smoothly excises a shell of thickness M^h about the Fermi surface. The ultraviolet end of the system is finite. Nevertheless, for convenience, an ultraviolet cutoff is also included.

The generating functional at cutoff h is

$$\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}};r,h) \;=\; \log \tfrac{1}{\mathcal{Z}_\Delta} \int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\; e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\int đk\,\bar{\mathbf{\Psi}}(k)(D-r\Delta)\mathbf{\Psi}(k)}\, d\mu_\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}};h) \tag{II.10}$$

where $d\mu_\Delta(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}};h)$ is the Grassmann-Gaussian measure with the regularized covariance (II.9). By a slight generalization of [1, Lemma 5], $\mathcal{S}_R(\boldsymbol{\Phi},\bar{\boldsymbol{\Phi}};r,h)$ exists. That is, for all λ in an h-dependent disk around zero, each of the connected Euclidean Green's functions

$$\prod_{i=1}^{p}\frac{\delta}{\delta\boldsymbol{\Phi}_{j_i}(\xi_i)}\ \mathcal{S}_R(\boldsymbol{\Phi},\bar{\boldsymbol{\Phi}};r,h)\ \prod_{i=1}^{p}\frac{\delta}{\delta\bar{\boldsymbol{\Phi}}_{\bar{\jmath}_i}(\bar{\xi}_i)}$$

is a well-defined distribution on $\mathbb{R}^{2p(d+1)}$. In particular, the coefficients $D_1(\lambda,\mu,\Delta;r,h)$ and $D_2(\lambda,\mu,\Delta;r,h)$ of the counterterms are convergent power series in λ uniquely determined by the renormalization condition

$$\boldsymbol{\Sigma}(0,|\mathbf{k}|=k_F;r,h)\ =\ 0$$

Our goal is to rigorously control the limit

$$\lim_{r\to0+}\ \lim_{h\to-\infty}\ \mathcal{S}_R(\boldsymbol{\Phi},\bar{\boldsymbol{\Phi}};r,h)$$

subject to the constraint

$$\begin{aligned}D_1(\lambda,\mu,\Delta;r,h)&=\Delta_1\\ D_2(\lambda,\mu,\Delta;r,h)&=\Delta_2\end{aligned}$$

and construct the many Fermion system (I.1). An important ingredient will be Ward identities for $\mathcal{S}_R(\boldsymbol{\Phi},\bar{\boldsymbol{\Phi}};r,h)$.

We now derive the simplest Ward identity for $\mathcal{S}_R(\boldsymbol{\Phi},\bar{\boldsymbol{\Phi}};r,h)$ using the global number symmetry. The action at cutoff h is

$$\mathcal{A}_R(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}};r,h)=-\mathcal{V}(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}})-\int đk\ \bar{\boldsymbol{\Psi}}_k(D(\lambda,\mu,\Delta)-r\Delta)\boldsymbol{\Psi}_k-\int đk\ \bar{\boldsymbol{\Psi}}_k\big(\tfrac{ik_0e(\mathbf{k})\sigma^3-\Delta}{\rho_h(k)}\big)\boldsymbol{\Psi}_k$$

Of course, $\frac{1}{\rho_h(k)}$ is infinite outside the support of $\rho_h(k)$. We only use this action to motivate a Ward identity in which

$$-\int đk\ \bar{\boldsymbol{\Psi}}_k\big(\tfrac{ik_0e(\mathbf{k})\sigma^3-\Delta}{\rho_h(k)}\big)\boldsymbol{\Psi}_k$$

is absorbed into the well-defined Fermionic-Gaussian measure $d\mu_\Delta(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}};h)$.

Now, under the number symmetry,

$$\begin{aligned}\boldsymbol{\Psi}(k)&\mapsto e^{i\theta\sigma^3}\boldsymbol{\Psi}(k)\\ \bar{\boldsymbol{\Psi}}(k)&\mapsto\bar{\boldsymbol{\Psi}}(k)e^{-i\theta\sigma^3}\end{aligned}$$

the action

$$\mathcal{A}_R(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}};r,h)\ \mapsto\ \mathcal{A}_R(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}};r,h)-\int đk\ \bar{\boldsymbol{\Psi}}(k)(R(2\theta)-\mathbb{1})(D-r\Delta-\tfrac{\Delta}{\rho_h(k)})\boldsymbol{\Psi}(k)$$

and

$$[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]\mapsto[\bar{\mathbf{\Phi}}e^{i\theta\sigma^3}\mathbf{\Psi}+\bar{\mathbf{\Psi}}e^{-i\theta\sigma^3}\mathbf{\Phi}]$$

To accomodate the new terms generated by applying the number symmetry to the action, we generalize (retaining the same name) the generating functional by allowing the external field to be a self-adjoint traceless matrix valued function $J(k)=J_1(k)\sigma^1+J_2(k)\sigma^2+J_3(k)\sigma^3$ of k. Namely,

$$\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,h)=\log\tfrac{1}{\mathcal{Z}_\Delta}\int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\,e^{-\mathcal{V}-\int d\!\!\bar{}k\,\bar{\mathbf{\Psi}}(k)\,(D-r\Delta-J(k))\mathbf{\Psi}(k)}\,d\mu_{\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}};h)}$$

Formally, the simplest Ward identity is

$$\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}},0;r,h)\ =\ \mathcal{S}_R(e^{-i\theta\sigma^3}\mathbf{\Phi},\bar{\mathbf{\Phi}}e^{i\theta\sigma^3},I(\theta;h);r,h)\tag{II.11a}$$

where

$$\begin{aligned}I(\theta;h)=&(D_1-r\Delta_1-\tfrac{\Delta_1}{\rho_h(k)})\big(1-\cos2\theta,-\sin2\theta,0\big)\cdot\sigma\\&+(D_2-r\Delta_2-\tfrac{\Delta_2}{\rho_h(k)})\big(\sin2\theta,1-\cos2\theta,0\big)\cdot\sigma\end{aligned}\tag{II.11b}$$

Again, the factor $1/\rho_h(k)$ in $I(\theta;h)$ is infinite outside the support of $\rho_h(k)$. However, integration against $d\mu_{\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}};h)}$ produces compensating factors of $\rho_h(k)$. For example, integrating by parts,

$$\begin{aligned}&\int d\mu_{\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}};h)}\Big(\int d\!\!\bar{}k\ \bar{\mathbf{\Psi}}(k)\,I(\theta;h)\mathbf{\Psi}(k)\Big)\ F(\mathbf{\Psi},\bar{\mathbf{\Psi}})\\&=\ -\int d\mu_{\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}};h)}\ \mathrm{tr}\int d\!\!\bar{}k\ \mathbf{C}_\Delta(k)\,\rho_h(k)^2I(\theta;h)\,\mathbf{C}_\Delta(k)\frac{\delta}{\delta\bar{\mathbf{\Psi}}(k)}F(\mathbf{\Psi},\bar{\mathbf{\Psi}})\frac{\delta}{\delta\mathbf{\Psi}(k)}\\&\quad-\ \mathrm{tr}\int d\!\!\bar{}k\ \mathbf{C}_\Delta(k)\,\rho_h(k)I(\theta;h)\ \times\int d\mu_{\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}};h)}\ F(\mathbf{\Psi},\bar{\mathbf{\Psi}})\end{aligned}$$

The odd looking factor $1/\rho_h(k)$ in $I(\theta;h)$ is an artifact of the smooth cutoff $\rho_h(k)$, which was chosen for technical reasons in [1,2,5,6]. A smooth cutoff is particularly useful when estimating renormalized diagrams. If periodic boundary conditions are imposed to regularize the infrared singularity, one obtains a, perhaps more natural looking, Ward identity. Let $L\gg1$ and define

$$\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,L)\ =\ \log\tfrac{1}{\mathcal{Z}_\Delta}\int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\,e^{-\mathcal{V}-\Sigma\frac{1}{L^{d+1}}\ \bar{\mathbf{\Psi}}(k)\,(D-r\Delta-J)\mathbf{\Psi}(k)}\,d\mu_{\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}};L)}$$

where the Grassmann-Gaussian measure

$$d\mu_{\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}};L)}\ =\ \tfrac{1}{\mathcal{Z}_0}\exp\Big\{-\sum_k\frac{1}{L^{d+1}}\ \bar{\mathbf{\Psi}}(k)\big(ik_0e(\mathbf{k})\sigma^3-\Delta\big)\mathbf{\Psi}(k)\Big\}\prod_{k,i}d\mathbf{\Psi}_i(k)\,d\bar{\mathbf{\Psi}}_i(k)$$

All sums and products over k are restricted to those points of the dual lattice $\frac{2\pi}{L}\mathbb{Z}^{d+1}$ that lie in the shell $0<k_0^2+e(\mathbf{k})^2\le1$. The corresponding Ward identity is

$$\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}},0;r,L)\ =\ \mathcal{S}_R(e^{-i\theta\sigma^3}\mathbf{\Phi},\bar{\mathbf{\Phi}}e^{i\theta\sigma^3},I(\theta;L);r,L)$$

where now

$$\begin{aligned} I(\theta;L) =&(D_1 - r\Delta_1 - \Delta_1)\big(1-\cos 2\theta, -\sin 2\theta, 0\big)\cdot\sigma \\ &+(D_2 - r\Delta_2 - \Delta_2)\big(\sin 2\theta, 1-\cos 2\theta, 0\big)\cdot\sigma \end{aligned}$$

We next give a sample application of the Ward identity derived above. The renormalized four point tensor $S_4{}^{i_1}{}_{i_2\,i_3}{}^{i_4}(s,t,q;J,h)$ is defined by

$$S_4{}^{i_1}{}_{i_2\,i_3}{}^{i_4}(k_1,k_2,k_3,k_4,J;r,h) \;=\; S_4{}^{i_1}{}_{i_2\,i_3}{}^{i_4}(s,t,q,J;r,h)\;(2\pi)^{d+1}\delta(k_1-k_2-k_3+k_4)$$

where

$$\begin{aligned} s &= \tfrac{1}{2}(k_3+k_4) \\ t &= \tfrac{1}{2}(k_1+k_2) \\ q &= k_3-k_4 \end{aligned}$$

and

$$S_4{}^{i_1}{}_{i_2\,i_3}{}^{i_4}(\xi_1,\xi_2,\xi_3,\xi_4;J,h) = \tfrac{\delta^2}{\delta\bar{\mathbf{\Phi}}_{i_1}(\xi_1)\delta\bar{\mathbf{\Phi}}_{i_4}(\xi_4)}\,\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,h)\,\tfrac{\delta^2}{\delta\mathbf{\Phi}_{i_2}(\xi_2)\delta\mathbf{\Phi}_{i_3}(\xi_3)}\,\Big|_{\mathbf{\Phi}=\bar{\mathbf{\Phi}}=0}$$

We show that some components of the four point tensor $S_4{}^{i_1}{}_{i_2\,i_3}{}^{i_4}(s,t,q;J,h)$ necessarily develop a singularity at $q=0$ in the limit $h\to-\infty$ and then $J\to 0$. Differentiating the Ward identity (II.11) with respect to θ and then setting $\theta=0$ yields

$$\Big\langle i[\bar{\mathbf{\Phi}}\sigma^3\mathbf{\Psi} - \bar{\mathbf{\Psi}}\sigma^3\mathbf{\Phi}] + \int đk\;\bar{\mathbf{\Psi}}(k)\,\frac{\partial J}{\partial\theta}(0;h)\mathbf{\Psi}(k)\Big\rangle_{r,h} = 0$$

where the expectation symbol

$$\langle F\rangle_{r,h} = \frac{\int F\;e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\int đk\;\bar{\mathbf{\Psi}}(k)\,(D-r\Delta)\mathbf{\Psi}(k)}\,d\mu_{\Delta}(\mathbf{\Psi},\bar{\mathbf{\Psi}};h)}{\int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\int đk\;\bar{\mathbf{\Psi}}(k)\,(D-r\Delta)\mathbf{\Psi}(k)}\,d\mu_{\Delta}(\mathbf{\Psi},\bar{\mathbf{\Psi}};h)}$$

Note that the expectation $\langle F\rangle_{r,h}$ depends on the external fields $\mathbf{\Phi},\bar{\mathbf{\Phi}}$.

For any $\gamma=\gamma_1\sigma^1+\gamma_2\sigma^2+\gamma_3\sigma^3$ set

$$\gamma^\sharp = \tfrac{i}{2}\,[\gamma,\sigma^3] = \gamma_1\sigma^2-\gamma_2\sigma^1 \tag{II.12}$$

In the event that $\gamma=\gamma_1\sigma^1+\gamma_2\sigma^2$

$$\gamma^\sharp = -i\sigma^3\gamma$$

Evaluating the derivative of I

$$\begin{aligned} \frac{\partial I}{\partial\theta}(0;h) &= (D_1 - r\Delta_1 - \tfrac{\Delta_1}{\rho_h(k)})(0,-2,0)\cdot\sigma + (D_2 - r\Delta_2 - \tfrac{\Delta_2}{\rho_h(k)})(2,0,0)\cdot\sigma \\ &= 2(r-1+\tfrac{1}{\rho_h(k)})\Delta^{\#} \end{aligned}$$

provided the BCS constraints are fulfilled. Substituting

$$2r\left\langle \int đk\ \bar{\mathbf{\Psi}}(k)\Delta^{\#}\mathbf{\Psi}(k)\right\rangle_{r,h} = \left\langle i[\bar{\mathbf{\Psi}}\sigma^3\mathbf{\Phi} - \bar{\mathbf{\Phi}}\sigma^3\mathbf{\Psi}] + 2\int đk\ \bar{\mathbf{\Psi}}(k)(1-\tfrac{1}{\rho_h(k)})\Delta^{\#}\mathbf{\Psi}(k)\right\rangle_{r,h} \tag{II.13}$$

Next, differentiate (II.13) with respect to $\bar{\mathbf{\Phi}}(p_1)$ on the left and with respect to $\mathbf{\Phi}(p_2)$ on the right and set $\mathbf{\Phi} = \bar{\mathbf{\Phi}} = 0$.

$$\begin{aligned} i\left[\left\langle \mathbf{\Psi}(p_1);\bar{\mathbf{\Psi}}(p_2)\right\rangle_{0,r,h}, \sigma^3\right] &= 2r\left\langle \mathbf{\Psi}(p_1)\bar{\mathbf{\Psi}}(p_2)\,;\int đk\,\bar{\mathbf{\Psi}}(k)\Delta^{\sharp}\mathbf{\Psi}(k)\right\rangle_{0,r,h} \\ &\quad + 2\int đk\,\tfrac{1-\rho_h(k)}{\rho_h(k)}\left\langle \mathbf{\Psi}(p_1)\bar{\mathbf{\Psi}}(p_2)\,;\,\bar{\mathbf{\Psi}}(k)\Delta^{\sharp}\mathbf{\Psi}(k)\right\rangle_{0,r,h} \end{aligned}$$

The notation $\langle F\rangle_{0,r,h}$ designates that $\langle F\rangle_{r,h}$ is evaluated at $\mathbf{\Phi} = \bar{\mathbf{\Phi}} = 0$. The truncated expectation value

$$\langle F;G\rangle_{r,h} = \langle FG\rangle_{r,h} - \langle F\rangle_{r,h}\langle G\rangle_{r,h}$$

Let F and G be monomials in the fields. The subtraction built into $\langle F;G\rangle_{r,h}$ "connects" F to G. Diagramatically, the fields in F and G are represented by ingoing and outgoing half-propagators. The Feynman diagrams contributing to the truncated expectation $\langle F;G\rangle_{r,h}$ have the property that every half propagator of F is connected to some half propagator of G and vice versa. By contrast to the fully connected expectation S_4, these graphs need not be connected.

By definition,

$$\begin{aligned} \lim_{r\to 0+}\lim_{h\to -\infty}\left[\left\langle \mathbf{\Psi}(p_1);\bar{\mathbf{\Psi}}(p_2)\right\rangle_{r,h}, \sigma^3\right] &= \left[S_2(p_1,p_2)\,,\,\sigma^3\right] \\ &= \left[S(p_1)\,,\,\sigma^3\right](2\pi)^{d+1}\delta(p_1-p_2) \end{aligned}$$

Restricting p_1 and p_2 to the Fermi surface, assuming that the number symmetry is broken (that is, $\Delta \neq 0$) and recalling the renormalization condition (II.4)

$$\begin{aligned} \lim_{r\to 0+}\lim_{h\to -\infty}\left[\left\langle \mathbf{\Psi}(p_1');\bar{\mathbf{\Psi}}(p_2')\right\rangle_{r,h}, \sigma^3\right] &= \left[C_\Delta(p_1')\,,\,\sigma^3\right](2\pi)^{d+1}\delta(p_1'-p_2') \\ &= 2i\frac{\Delta^\sharp}{\Delta^2}(2\pi)^{d+1}\delta(p_1'-p_2') \\ &\neq 0 \end{aligned}$$

where

$$p' = (0, \tfrac{\mathbf{p}}{|\mathbf{p}|}\sqrt{2m\mu})$$

In the event that the limit

$$\lim_{r\to 0+}\lim_{h\to -\infty}\left\langle \mathbf{\Psi}(p_1')\bar{\mathbf{\Psi}}(p_2')\,;\int đk\,\bar{\mathbf{\Psi}}(k)\Delta^{\sharp}\mathbf{\Psi}(k)\right\rangle_{r,h}$$

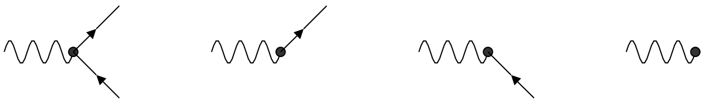
exists and is finite, we conclude that

$$-\frac{\Delta^\sharp}{\Delta^2}(2\pi)^{d+1}\delta(p_1'-p_2') = \lim_{r\to 0+}\lim_{h\to-\infty}\int đk\, \tfrac{1-\rho_h(k)}{\rho_h(k)} \left\langle \boldsymbol{\Psi}(p_1)\bar{\boldsymbol{\Psi}}(p_2)\,;\, \bar{\boldsymbol{\Psi}}(k)\Delta^\sharp\boldsymbol{\Psi}(k)\right\rangle_{r,h}$$

Assuming that the right hand side approaches zero as the ultraviolet cutoff is removed, we get a contradiction for all sufficiently large ultraviolet cutoffs. So, the limit cannot be finite. See [9, (8-115)] for a similar result.

§III String and Ladder Amputation

In this section we study the topology of connected graphs constructed from $2n$ vertices $\bullet$, n interaction lines and $n-e$ particle lines . Each vertex is attached to one interaction line, at most one incoming particle line and at most one outgoing particle line. Thus, there are four possible configurations at a vertex:



By definition, a vertex has $1-\alpha$ "incoming external particle legs" and $1-\beta$ "outgoing external particle legs" when it is attached to α incoming particle lines and β outgoing particle lines. External legs are not part of the graph. However, it is convenient to draw them in to aid enumerating them. Our goal is to decompose each of these graphs having $e\ge 6$ external particle legs into an amputated general vertex and a set of general ladders and a set of general strings. When $e=4$ the graph is expressed as a general ladder with two particle irreducible rungs joined by pairs of strings.

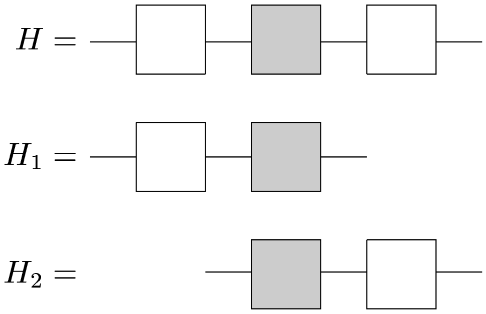
The first step is a topological classification of all pairs of intersecting four legged subdiagrams. A subdiagram H of G is determined by a set of one or more particle and/or interaction lines of G. By definition H is the union of the lines and the vertices at the ends of the lines. A line of $G\setminus H$ is an external line for H if it is attached to a vertex of H. An external leg of G, say at vertex v, is also an external leg of H if $v\in H$. Two subgraphs H_1 and H_2 **overlap** if

$$H_1\cap H_2\neq\emptyset \quad H_1\not\subset H_2 \quad H_2\not\subset H_1$$

Lemma III.1 *Let H be a connected graph which is the union of two overlapping subgraphs H_1 and H_2. If H_1 and H_2 each have two external particle legs and no external interaction legs then H must be of one of the two following forms:*

If H is not a vacuum graph, the second form is ruled out.

The first form is read as



or with H_1 and H_2 exchanged.

Proof: Any external line of $H_1 \cap H_2$ must be an external line of H_1 or of H_2 or of both. Hence all external lines of $H_1 \cap H_2$ must be particle lines. Let

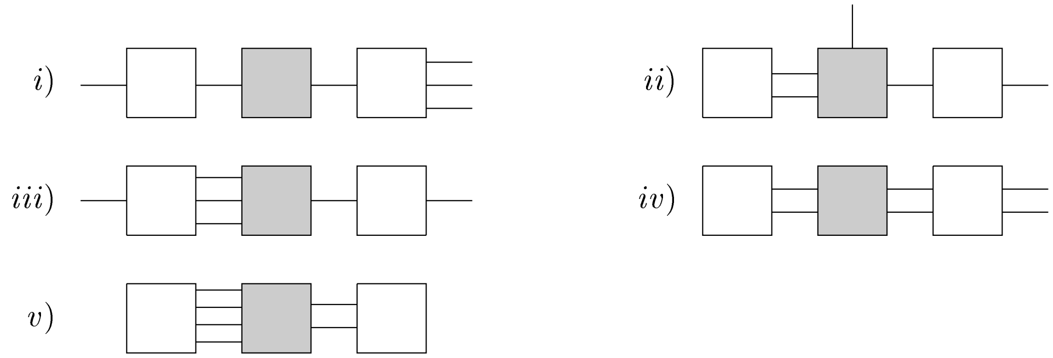
$$\begin{aligned}
n_1 &= \text{the number of external lines of } H_1 \cap H_2 \text{ belonging to } H_1 \\
n_2 &= \text{the number of external lines of } H_1 \cap H_2 \text{ belonging to } H_2 \\
n_\emptyset &= \text{the number of external lines of } H_1 \cap H_2 \text{ not in } H
\end{aligned}$$

Since $H_1 \not\subset H_1 \cap H_2$ and H is connected, $n_1 \geq 1$. Since $H_2 \not\subset H_1 \cap H_2$, $n_2 \geq 1$. Since H_2 and H_1 have two external legs, $n_1 + n_\emptyset \leq 2$ and $n_2 + n_\emptyset \leq 2$. As $n_1 + n_2 + n_\emptyset$ must be even, we must have $n_1 = n_2 = 1$, $n_\emptyset = 0$ or $n_1 = n_2 = 2$, $n_\emptyset = 0$. ■

Corollary III.2 *Let G be a connected graph with external legs $\ell_1, \cdots, \ell_e$, $e \geq 4$. For each $1 \leq i \leq e$ either there is no two legged subgraph of G having $\boldsymbol{\ell}_i$ as an external leg, or there is a unique maximal two legged subgraph S_i of G with $\boldsymbol{\ell}_i$ as an external leg. Furthermore $S_i \cap S_j = \emptyset$ for all $i \neq j$.*

By definition the string-amputation of a connected graph G having four or more legs is gotten by deleting each S_i together with the line of G that is an external line of S_i.

Lemma III.3 *Let H be a connected graph which is the union of two overlapping subgraphs H_1 and H_2. The subgraph H_2 need not be connected. If H_1 and H_2 have two and four external particle legs respectively and no external interaction legs then H must be of one of the following five forms:*



Proof: As in the proof of Lemma III.1,

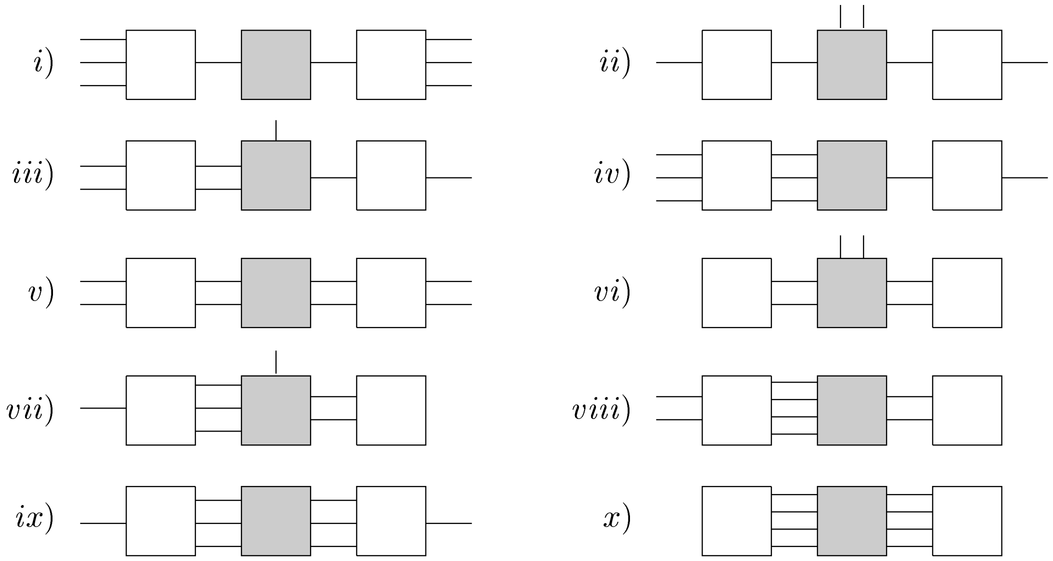
$$\begin{array}{ll} n_1 \geq 1 & n_2 \geq 1 \\ n_1 + n_\emptyset \leq 2 & n_2 + n_\emptyset \leq 4 \\ \multicolumn{2}{c}{n_1 + n_2 + n_\emptyset \in 2\mathbb{Z}} \end{array}$$

Consequently, $(n_1, n_2, n_\emptyset)$ must be one of

n_1	1	1	1	2	2
n_2	1	2	3	2	4
$n_\emptyset$	0	1	0	0	0
form	*(i)*	*(ii)*	*(iii)*	*(iv)*	*(v)*

For example, $(n_1, n_2, n_\emptyset) = (1, 3, 0)$ implies that the subgraph H_1 and H_2 each one external leg which is external to H. This gives form *(iii)*. ■

Lemma III.4 *Let H be a connected graph which is the union of two overlapping subgraphs H_1 and H_2. That is $H = H_1 \cup H_2$, $H_1 \cap H_2 \neq \emptyset$, but $H_1 \not\subset H_2$ and $H_2 \not\subset H_1$. Note that H_1 and H_2 need not be connected. If H_1 and H_2 each have four external particle legs and no external interaction legs then H must be of one of the following ten forms or their reflections about the central shaded square:*



Proof: As in the proof of Lemma III.1,

$$\begin{array}{cc} n_1 \geq 1 & n_2 \geq 1 \\ n_1 + n_\emptyset \leq 4 & n_2 + n_\emptyset \leq 4 \\ \multicolumn{2}{c}{n_1 + n_2 + n_\emptyset \in 2\mathbb{Z}} \end{array}$$

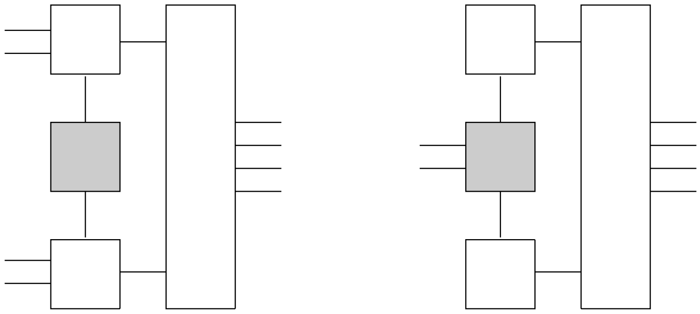
We have the table, for $n_1 \geq n_2$,

n_1	1	1	2	3	2	2	3	4	3	4
n_2	1	1	1	1	2	2	2	2	3	4
$n_\emptyset$	0	2	1	0	0	2	1	0	0	0
form	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	$(viii)$	(ix)	(x)

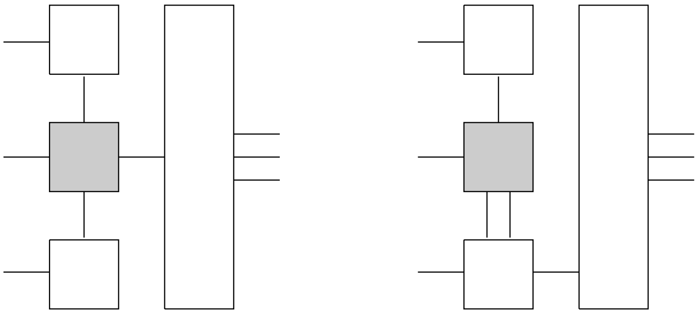
■

In preparation for amputating generalized ladders from many legged graphs we prove

Theorem III.5 *Let G be any string-amputated connected graph with $e \geq 6$ external particle legs and no external interaction legs. Let G_1 and G_2 be overlapping four legged subgraphs of G without external interaction legs. Furthermore, assume that G_1 and G_2 each have precisely two legs external to G. Then we have one of the two possible configurations:*



Proof: Define $H = G_1 \cup G_2,\ H_1 = G_1,\ H_2 = G_2$. As G is connected and string-amputated, G_1 and G_2 and consequently H are necessarily connected. It now suffices to apply Lemma III.4 and then connect H into the rest of G in all possible ways. The two permissible configurations arise from Lemma III.4 forms (i) and (ii). Forms *(iv), (vii), (viii), (ix)* and *(x)* are ruled out by the requirement that G_2 have two legs external to G. For forms (v) and (vi), if G_1 and G_2 were each have two legs external to G, then G would have a total of four and two external legs, respectively. The configurations

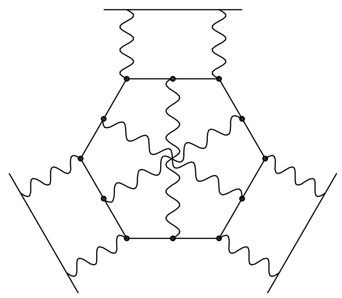


arising from forms (ii) and (iii) are not string amputated. ■

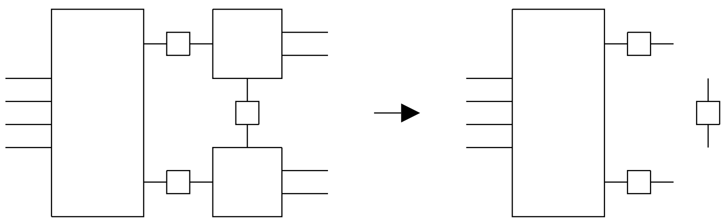
Corollary III.6 *Let G be a string-amputated connected graph with external particle legs $\ell_1,\ \cdots,\ \ell_e\ ,\ e \geq 6$ and no external interaction legs. Define $\ell_i \sim \ell_j$ if $i = j$ or if ℓ_i, ℓ_j are the external legs of a four legged subgraph of G which has exactly two globally external legs. Then $\sim$ is an equivalence relation and the corresponding equivalence classes have one or two elements.*

Proof: We show that if $i,\ j$ and k are three distinct indices then either $i \not\sim j$ or $i \not\sim k$. Otherwise, there would have to be two overlapping four-legged subdiagrams, with one of the subdiagrams having ℓ_i and ℓ_j as its globally external legs and the other having ℓ_i and ℓ_k as its globally external legs. This contradicts Theorem III.5. ■

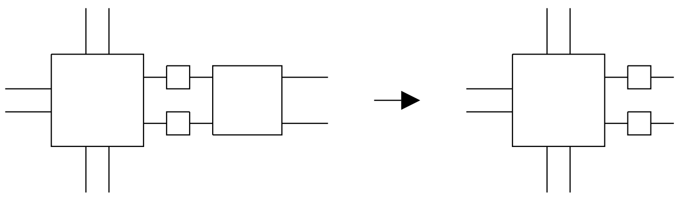
The graph below contains no four legged subdiagram with precisely two globally external legs. Hence its equivalence classes are all singlets.



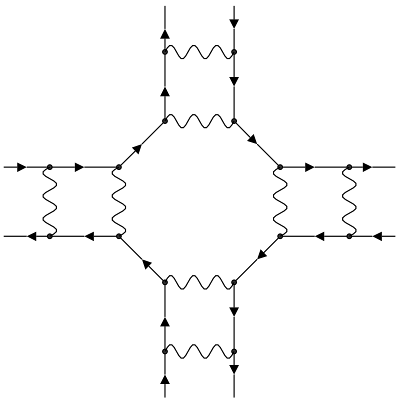
We define a connected graph to be one particle irreducible (1PI) if it cannot be disconnected by cutting one **particle** line. Let G be a string amputated, connected graph with $e \geq 6$ external legs. Then by Theorem III.5, there is, for each pair of distinct equivalent external legs a unique maximal four legged 1PI subgraph whose globally external legs are the given pair. Furthermore, the 1PI subdiagrams corresponding to different equivalence classes are disjoint. By definition the **ladder amputation** of G is gotten by removing all of these 1PI subdiagrams. For example, under ladder amputation

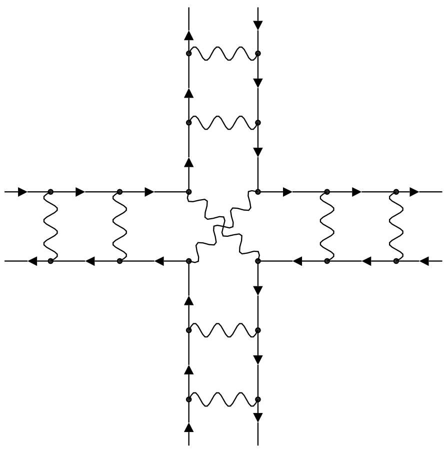


and



Thus the ladder amputation of G need not be connected. For example

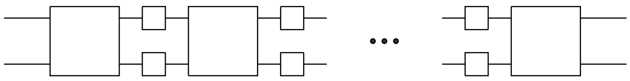




We now take a closer look at the structure of the "ladder" subgraphs that are removed in the above amputation process. Each such subgraph is four-legged, connected and string-amputated. Furthermore, the four external legs are paired into two incoming and two outgoing legs. Such a diagram is said to be **channel 2PI** if it is impossible to disconnect the pair of incoming external legs from the pair of outgoing external legs by cutting at most two **particle** lines.

Let G be a connected, string-amputated four-legged graph with two of its external legs designated as incoming and the other two designated as outgoing. Let H_1 and H_2 be four-legged subgraphs of G and suppose that G's incoming external legs are also external legs for H_1 and H_2. We claim that, if $H_1 \neq H_2$, then H_1 and H_2 cannot both be channel 2PI. If, for example, H_1 is properly contained in H_2, then H_2 is disconnected by cutting the internal line(s) of H_2 that are outgoing external lines of H_1. If H_1 and H_2 overlap, then only form *ii*) of Lemma III.4 is possible and neither H_1 and H_2 are channel 2PI. Thus there is a unique channel 2PI subgraph of G containing G's incoming legs. Remove this subgraph and string-amputate the result. This leaves another connected, string-amputated four-legged graph. Iterating, we get

Theorem III.7 *Let G be a connected, string-amputated four-legged graph with two of its external legs designated as incoming and the other two designated as outgoing. Then G has a unique decomposition*



Here, the four legged boxes are channel 2PI.

The results of this section will be used in §IV to interpret the Goldstone Boson propagator graphically.

§IV The Ward Identity

We now use a local gauge transformation to derive a more general Ward identity. Let P be a general propagator and define the action, $\mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,P)$ by

$$\mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,P) \;=\; \mathcal{A}_R^{\rm int}(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r) - \int đk\; \bar{\mathbf{\Psi}}_k P^{-1}\mathbf{\Psi}_k$$

where

$$\mathcal{A}_R^{\rm int}(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r) \;=\; [\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}] - \mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}}) - \int d\xi\; \bar{\mathbf{\Psi}}(\xi)(Dr\Delta J(\xi))\mathbf{\Psi}(\xi)$$

and

$$\begin{aligned} \Delta &= \Delta_1\sigma^1+\Delta_2\sigma^2 \\ D &= D_1\sigma^1+D_2\sigma^2+D_3\sigma^3 \\ J(\xi) &= J_1(\xi)\sigma^1+J_2(\xi)\sigma^2+J_3(\xi)\sigma^3 \end{aligned}$$

The interaction $\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})$ is defined in (II.3).

If $P=\rho_h C_\Delta$ with cutoff ρ_h (see (II.9)), then the counterterm $D = D(\lambda,\mu,\Delta;r,h)$ in

$$\mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,h) = \mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,\rho_h C_\Delta)$$

is determined by the renormalization condition

$$\Sigma(0,|\mathbf{k}|=k_F;r,h,\Delta)=0$$

The generating functional corresponding to the action $\mathcal{A}_R$ is

$$\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,P) = \log\tfrac{1}{Z}\int e^{\mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,P)} \prod_{k,i} d\mathbf{\Psi}^i(k)\, d\bar{\mathbf{\Psi}}_i(k)$$

There is some redundancy in this generating functional. The expectation value of $F(\mathbf{\Psi},\bar{\mathbf{\Psi}})$ is

$$\big\langle F(\mathbf{\Psi},\bar{\mathbf{\Psi}})\big\rangle = \frac{\int F(\mathbf{\Psi},\bar{\mathbf{\Psi}}) e^{\mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,P)}\;\prod_{k,i} d\mathbf{\Psi}^i(k)\, d\bar{\mathbf{\Psi}}_i(k)}{\int e^{\mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,P)}\;\prod_{k,i} d\mathbf{\Psi}^i(k)\, d\bar{\mathbf{\Psi}}_i(k)}\;.$$

Differentiating

$$(2\pi)^{d+1}\frac{\delta}{\delta\bar{\mathbf{\Phi}}_i(k)}\,\mathcal{S}_R = \big\langle \mathbf{\Psi}^i(k)\big\rangle$$

and

$$\begin{aligned} (2\pi)^{2(d+1)}\frac{\delta}{\delta\bar{\mathbf{\Phi}}_i(k)}\,\mathcal{S}_R\,\frac{\delta}{\delta\mathbf{\Phi}^j(k)} &= \big\langle \mathbf{\Psi}^i(k)\bar{\mathbf{\Psi}}_j(k)\big\rangle - \big\langle \mathbf{\Psi}^i(k)\big\rangle\big\langle\bar{\mathbf{\Psi}}_j(k)\big\rangle \\ &= \big\langle \mathbf{\Psi}^i(k);\bar{\mathbf{\Psi}}_j(k)\big\rangle \end{aligned}$$

where the truncated expectation value

$$\langle F;G\rangle = \langle FG\rangle - \langle F\rangle\langle G\rangle$$

We have

$$\begin{aligned} \frac{\delta}{\delta J_\mu(k)}\mathcal{S}_R &= (2\pi)^{d+1}\sum_{i,j}(\sigma^\mu)^j{}_i\Big[-\tfrac{\delta}{\delta\bar{\mathbf{\Phi}}_i(k)}\,\mathcal{S}_R\,\tfrac{\delta}{\delta\mathbf{\Phi}^j(k)} - \Big(\tfrac{\delta}{\delta\bar{\mathbf{\Phi}}_i(k)}\,\mathcal{S}_R\Big)\Big(\mathcal{S}_R\,\tfrac{\delta}{\delta\mathbf{\Phi}^j(k)}\Big)\Big] \\ &= -(2\pi)^{d+1}\;{\rm tr}\,\sigma^\mu\Big[\tfrac{\delta}{\delta\bar{\mathbf{\Phi}}(k)}\,\mathcal{S}_R\,\tfrac{\delta}{\delta\mathbf{\Phi}(k)} + \Big(\tfrac{\delta}{\delta\bar{\mathbf{\Phi}}(k)}\,\mathcal{S}_R\Big)\Big(\mathcal{S}_R\,\tfrac{\delta}{\delta\mathbf{\Phi}(k)}\Big)\Big] \end{aligned}$$

Theorem IV.1 (Ward Identity) *Suppose that the interaction*

$$\langle k_1, k_2|\mathrm{V}|k_3, k_4\rangle \;=\; \widehat{V}(k_1 k_3)$$

on the support of the delta function $\delta_{(k_1+k_2-k_3-k_4)}$. *Let* $\theta(\xi)$ *be a smooth function on* $\mathbb{R}^{d+1}$. *Then*

$$\begin{aligned}
&\mathcal{S}_R(\boldsymbol{\Phi}, \bar{\boldsymbol{\Phi}}, J; r, \rho_h C_\Delta)\\
&\quad = \mathcal{S}_R\Big(e^{-i\theta(\xi)\sigma^3}\boldsymbol{\Phi}\ ,\ \bar{\boldsymbol{\Phi}}e^{i\theta(\xi)\sigma^3}\ ,\ e^{-i\theta(\xi)\sigma^3}J(\xi)e^{i\theta(\xi)\sigma^3} + I(\theta;h)\ ;\ r\ ,\ \rho_h C_\Delta + P(\theta,h)\Big)
\end{aligned}$$

where

$$\begin{aligned}
I(\theta;h) &= (1-r)\Big(1\!\!1 - R\big(2\theta(\xi)\big)\Big)\Delta\\
&= \Delta_1(1-r)\big(1-\cos 2\theta(\xi), -\sin 2\theta(\xi), 0\big)\cdot\sigma\\
&\qquad + \Delta_2(1-r)\big(\sin 2\theta(\xi), 1-\cos 2\theta(\xi), 0\big)\cdot\sigma\\
P(\theta;h) &= \rho_h(k)\big(C_\Delta(\theta) - C_\Delta\big) + \Big(e^{-i\theta(\xi)\sigma^3}\rho_h(k)e^{i\theta(\xi)\sigma^3} - \rho_h(k)\Big)\,C_\Delta(\theta)\\
C_\Delta(\theta) &= e^{-i\theta(\xi)\sigma^3}C_\Delta e^{i\theta(\xi)\sigma^3}\\
&= \frac{1}{ik_0 - i\frac{\partial\theta}{\partial\tau}\sigma^3 - e\left(\mathbf{k}+\sigma^3\nabla\theta\right)\sigma^3 - R(2\theta(\xi))\Delta}
\end{aligned}$$

when the BCS constraints $D_1 = \Delta_1,\ D_2 = \Delta_2$ *are satisfied. An operator* $f(k)g(\xi)$ *is multiplication in position space by* $g(\xi)$ *followed by multiplication in momentum space by* $f(k)$.

Recall that

$$R(\theta)(v\cdot\sigma) = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}\cdot\sigma$$

Elementary manipulations yield

$$\begin{aligned}
P(\theta;h) &= \rho_h(k)C_\Delta(\theta)\left(i\frac{\partial\theta}{\partial\tau}\sigma^3 + \tfrac{1}{2m}\left(\mathbf{k}\cdot\nabla\theta + \nabla\theta\cdot\mathbf{k} + (\nabla\theta)^2\sigma^3\right) - (1\!\!1 - R(2\theta))\Delta\right)C_\Delta\\
&\qquad + \Big(e^{-i\theta(\xi)\sigma^3}\rho_h(k)e^{i\theta(\xi)\sigma^3} - \rho_h(k)\Big)\,C_\Delta(\theta)
\end{aligned}$$

Proof: To obtain the Ward identity we make the change of variables

$$\begin{aligned}
\boldsymbol{\Psi}(\xi) &= e^{i\theta(\xi)\sigma^3}\boldsymbol{\Psi}'(\xi)\\
\bar{\boldsymbol{\Psi}}(\xi) &= \bar{\boldsymbol{\Psi}}'(\xi)e^{-i\theta(\xi)\sigma^3}
\end{aligned}$$

and use

$$
\begin{aligned}
e^{-i\theta\sigma^3}\sigma^0 e^{i\theta\sigma^3} &= \sigma^0 \\
e^{-i\theta\sigma^3}\sigma^1 e^{i\theta\sigma^3} &= \cos 2\theta\ \sigma^1 + \sin 2\theta\ \sigma^2 \\
e^{-i\theta\sigma^3}\sigma^2 e^{i\theta\sigma^3} &= -\sin 2\theta\ \sigma^1 + \cos 2\theta\ \sigma^2 \\
e^{-i\theta\sigma^3}\sigma^3 e^{i\theta\sigma^3} &= \sigma^3
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathcal{V}(\boldsymbol{\Psi}, \bar{\boldsymbol{\Psi}}) &= \tfrac{\lambda}{2}\int d\xi_1 d\xi_2\ \big(\bar{\boldsymbol{\Psi}}(\xi_1)\sigma^3\boldsymbol{\Psi}(\xi_1)\big)\, V(\xi_1\xi_2)\, \big(\bar{\boldsymbol{\Psi}}(\xi_2)\sigma^3\boldsymbol{\Psi}(\xi_2)\big) \\
&= \mathcal{V}(\boldsymbol{\Psi}', \bar{\boldsymbol{\Psi}}')
\end{aligned}
$$

and

$$
\begin{aligned}
-\int đk\, \bar{\boldsymbol{\Psi}}_k(Dr\Delta)\boldsymbol{\Psi}_k &= -\int d\xi\ \bar{\boldsymbol{\Psi}}'(\xi)e^{-i\theta(\xi)\sigma^3}(Dr\Delta)e^{i\theta(\xi)\sigma^3}\boldsymbol{\Psi}'(\xi) \\
&= -\int d\xi\ \bar{\boldsymbol{\Psi}}'(\xi)(Dr\Delta)\boldsymbol{\Psi}'(\xi) - \int d\xi\ \bar{\boldsymbol{\Psi}}'(\xi)(R(2\theta)-\mathbb{1})(Dr\Delta)\boldsymbol{\Psi}'(\xi)
\end{aligned}
$$

The field $(\boldsymbol{\Psi}, \bar{\boldsymbol{\Psi}})$ is a Grassmann-Gaussian process with covariance $\rho_h(k)C_\Delta$ so that $(\boldsymbol{\Psi}', \bar{\boldsymbol{\Psi}}')$ is a Grassmann-Gaussian process with covariance $e^{-i\theta(\xi)\sigma^3}\rho_h(k)C_\Delta e^{i\theta(\xi)\sigma^3}$. Hence

$$
\begin{aligned}
P(\theta,h) &= e^{-i\theta(\xi)\sigma^3}\rho_h(k)C_\Delta e^{i\theta(\xi)\sigma^3} - \rho_h(k)C_\Delta \\
&= e^{-i\theta(\xi)\sigma^3}\rho_h(k)e^{i\theta(\xi)\sigma^3}C_\Delta(\theta) - \rho_h(k)C_\Delta \\
&= \rho_h(k)C_\Delta(\theta) - \rho_h(k)C_\Delta + \Big(e^{-i\theta(\xi)\sigma^3}\rho_h(k)e^{i\theta(\xi)\sigma^3} - \rho_h(k)\Big)\, C_\Delta(\theta)
\end{aligned}
$$

Subtracting

$$
\begin{aligned}
&\rho_h(k)C_\Delta(\theta) - \rho_h(k)C_\Delta \\
&\quad= \rho_h(k)C_\Delta(\theta)\left(i\frac{\partial\theta}{\partial\tau}\sigma^3 + e\left(\mathbf{k}+\sigma^3\nabla\theta\right)\sigma^3 - e\left(\mathbf{k}\right)\sigma^3 - (\mathbb{1}-R(2\theta))\Delta\right) C_\Delta \\
&\quad= \rho_h(k)C_\Delta(\theta)\left(i\frac{\partial\theta}{\partial\tau}\sigma^3 + \tfrac{1}{2m}\left(\mathbf{k}\cdot\nabla\theta + \nabla\theta\cdot\mathbf{k} + (\nabla\theta)^2\sigma^3\right) - (\mathbb{1}-R(2\theta))\Delta\right) C_\Delta
\end{aligned}
$$

■

Recall that, for $\gamma = \gamma_1\sigma^1 + \gamma_2\sigma^2 + \gamma_3\sigma^3$, the notation $\gamma^\#$ is defined in (II.12).

Proposition IV.2 *Suppose that the interaction*

$$
\langle k_1, k_2|\mathrm{V}|k_3, k_4\rangle = \widehat{V}(k_1 k_3)
$$

on the support of the delta function $\delta(k_1+k_2-k_3-k_4)$. *If the BCS constraints are satisfied, then*

$$
\begin{aligned}
&2r\left\langle \bar{\mathbf{\Psi}}(\xi)\Delta^{\#}\mathbf{\Psi}(\xi)\right\rangle - i\partial_0\left\langle \bar{\mathbf{\Psi}}(\xi)\sigma^3\mathbf{\Psi}(\xi)\right\rangle + \tfrac{i}{2m}\sum_{j=1}^{d}\partial_j\Big\langle (\partial_j\bar{\mathbf{\Psi}})(\xi)\mathbf{\Psi}(\xi) - \bar{\mathbf{\Psi}}(\xi)(\partial_j\mathbf{\Psi})(\xi)\Big\rangle\\
&\quad= i\left\langle \bar{\mathbf{\Psi}}(\xi)\sigma^3\mathbf{\Phi}(\xi)\right\rangle - i\left\langle \bar{\mathbf{\Phi}}(\xi)\sigma^3\mathbf{\Psi}(\xi)\right\rangle - 2\left\langle \bar{\mathbf{\Psi}}(\xi)J^{\#}(\xi)\mathbf{\Psi}(\xi)\right\rangle + U\\
&\qquad\quad + \text{ terms independent of } J, \mathbf{\Phi}, \bar{\mathbf{\Phi}}
\end{aligned}
$$

where

$$
\begin{aligned}
U &= \mathrm{tr}\,\Big(\Delta^{\#}\rho_h C_\Delta A C_\Delta(1-\rho_h) + (1-\rho_h)\Delta^{\sharp} C_\Delta A C_\Delta \rho_h\Big)(\xi,\xi)\\
&\quad - i\,\mathrm{tr}\,\Big(\sigma^3\rho_h C_\Delta A(1-\rho_h)C_\Delta(ik_0 e(\mathbf{k})\sigma^3) - \sigma^3(ik_0 e(\mathbf{k})\sigma^3)(1-\rho_h)C_\Delta A C_\Delta\rho_h\Big)(\xi,\xi)
\end{aligned}
$$

and

$$
A(\xi',\xi) = \Big\langle \Big(\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}\mathcal{A}_R^{\mathrm{int}}\Big)\Big(\mathcal{A}_R^{\mathrm{int}}\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}\Big)\Big\rangle + \Big\langle \tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}\mathcal{A}_R^{\mathrm{int}}\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}\Big\rangle
$$

In momentum space

$$
\begin{aligned}
&\Big\langle \int đk\ \bar{\mathbf{\Psi}}(k+q)\Big(2r\Delta^{\#} - q_0\sigma^3 + \tfrac{i}{2m}\mathbf{q}^2 + \tfrac{i}{m}\langle\mathbf{q},\mathbf{k}\rangle\Big)\mathbf{\Psi}(k)\Big\rangle\\
&= i\Big\langle \int đk\ \Big(\bar{\mathbf{\Psi}}(k+q)\sigma^3\mathbf{\Phi}(k) - \bar{\mathbf{\Phi}}(k+q)\sigma^3\mathbf{\Psi}(k)\Big)\Big\rangle - 2\Big\langle \int đk\, đp\ \bar{\mathbf{\Psi}}(k+p+q)\widetilde{J}(p)^{\#}\mathbf{\Psi}(k)\Big\rangle\\
&\qquad + \widetilde{U} + \text{ terms independent of } J, \mathbf{\Phi}, \bar{\mathbf{\Phi}}
\end{aligned}
$$

Proof: If $d\mu_{P(\varepsilon)}(\mathbf{\Psi},\bar{\mathbf{\Psi}})$ is any one parameter family of Grassmann-Gaussian measures, then

$$
\tfrac{d}{d\varepsilon}\int F(\mathbf{\Psi},\bar{\mathbf{\Psi}})d\mu_{P(\varepsilon)}(\mathbf{\Psi},\bar{\mathbf{\Psi}}) = -\int d\mu_{P(\varepsilon)}(\mathbf{\Psi},\bar{\mathbf{\Psi}})\int d\xi d\xi'\ \mathrm{tr}\ \tfrac{d}{d\varepsilon}P(\xi,\xi')\ \tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}F(\mathbf{\Psi},\bar{\mathbf{\Psi}})\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}
$$

Hence, for even elements F of the Grassmann algebra,

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\log\tfrac{1}{Z}\int e^{F(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\int d\xi d\xi'\,\bar{\mathbf{\Psi}}(\xi)P^{-1}(\xi,\xi';\varepsilon)\mathbf{\Psi}(\xi')}\prod_{k,i}d\mathbf{\Psi}^i(k)\,d\bar{\mathbf{\Psi}}_i(k)\\
&= \frac{d}{d\varepsilon}\log\tfrac{Z_0}{Z}\int e^{F(\mathbf{\Psi},\bar{\mathbf{\Psi}})}d\mu_{P(\varepsilon)}(\mathbf{\Psi},\bar{\mathbf{\Psi}})\\
&= -\tfrac{1}{\int e^F d\mu_{P(\varepsilon)}}\int d\mu_{P(\varepsilon)}e^F\int d\xi d\xi'\ \mathrm{tr}\ \tfrac{d}{d\varepsilon}P(\xi,\xi';\varepsilon)\Big\{\Big(\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}F\Big)\Big(F\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}\Big) + \tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}F\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}\Big\}\\
&\quad + \tfrac{Z}{Z_0}\tfrac{d}{d\varepsilon}\tfrac{Z_0}{Z}
\end{aligned}
$$

Now replace $\theta(\xi)$ by $\varepsilon\theta(\xi)$ in the Ward identity, differentiate with respect to ε and then set ε to zero. One obtains the derived Ward identity

$$
\begin{aligned}
0 &= -i\left\langle\left[\theta(\xi)\bar{\mathbf{\Psi}}(\xi)\sigma^3\mathbf{\Phi}(\xi)\right]\right\rangle + i\left\langle\left[\theta(\xi)\bar{\mathbf{\Phi}}(\xi)\sigma^3\mathbf{\Psi}(\xi)\right]\right\rangle\\
&\quad + 2\left\langle\left[\theta(\xi)\bar{\mathbf{\Psi}}(\xi)\theta(\xi)J^{\#}(\xi)\mathbf{\Psi}(\xi)\right]\right\rangle - 2(1-r)\left\langle\left[\theta(\xi)\bar{\mathbf{\Psi}}(\xi)\Delta^{\#}\mathbf{\Psi}(\xi)\right]\right\rangle\\
&\quad - \int d\xi d\xi'\ \mathrm{tr}\ \tfrac{d}{d\varepsilon}P(\xi,\xi';\varepsilon\theta)\Big|_{\varepsilon=0}\Big\{\Big\langle\Big(\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}\mathcal{A}_R^{\mathrm{int}}\Big)\Big(\mathcal{A}_R^{\mathrm{int}}\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}\Big)\Big\rangle + \Big\langle\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}\mathcal{A}_R^{\mathrm{int}}\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}\Big\rangle\Big\}\\
&\quad + \tfrac{Z}{Z_0}\tfrac{d}{d\varepsilon}\tfrac{Z_0}{Z}\Big|_{\varepsilon=0}
\end{aligned}
$$

We now evaluate the derivatives. Since

$$\mathcal{A}_R^{\rm int}(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},0;r) \;=\; \int d\eta\,\big(\bar{\mathbf{\Phi}}(\eta)\mathbf{\Psi}(\eta)+\bar{\mathbf{\Psi}}(\eta)\mathbf{\Phi}(\eta)\big)-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\int d\eta\,\bar{\mathbf{\Psi}}(\eta)\big(Dr\Delta J\big)\mathbf{\Psi}(\eta)$$

we have

$$\begin{aligned}
\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}\mathcal{A}_R^{\rm int} &= \mathbf{\Phi}(\xi')-\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\big(Dr\Delta J(\xi')\big)\mathbf{\Psi}(\xi')\\
\mathcal{A}_R^{\rm int}\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)} &= \bar{\mathbf{\Phi}}(\xi)-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}-\bar{\mathbf{\Psi}}(\xi)\big(Dr\Delta J(\xi)\big)\\
\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}\mathcal{A}_R^{\rm int}\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)} &= -\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}-\big(Dr\Delta J(\xi)\big)\delta(\xi-\xi')
\end{aligned}$$

The derivative of the propagator is

$$\begin{aligned}
\tfrac{d}{d\varepsilon}P(\xi,\xi';\varepsilon\theta)\big|_{\varepsilon=0} &= -i\rho_h\theta\sigma^3C_\Delta+i\rho_hC_\Delta\theta\sigma^3-i\theta\sigma^3\rho_hC_\Delta+i\rho_h\theta\sigma^3C_\Delta\\
&= -i\theta(\xi)\sigma^3(\rho_hC_\Delta)(\xi-\xi')+i(\rho_hC_\Delta)(\xi-\xi')\sigma^3\theta(\xi')
\end{aligned}$$

Recall that $\theta(\xi)$ is a multiplication operator in position space while $\rho_h(k)$ and $C_\Delta(k)$ are multiplication operators in momentum space.

Let

$$A(\xi',\xi)=\Big\langle\Big(\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}\mathcal{A}_R^{\rm int}\Big)\Big(\mathcal{A}_R^{\rm int}\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}\Big)\Big\rangle+\Big\langle\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}\mathcal{A}_R^{\rm int}\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}\Big\rangle$$

Then

$$\begin{aligned}
&-\int d\xi d\xi'\ {\rm tr}\,\tfrac{d}{d\varepsilon}P(\xi,\xi';\varepsilon\theta)\big|_{\varepsilon=0}\ A(\xi',\xi)\\
&= -{\rm tr}\int d\xi d\xi'\ \Big(-i\theta(\xi)\sigma^3(\rho_hC_\Delta)(\xi-\xi')+i(\rho_hC_\Delta)(\xi-\xi')\sigma^3\theta(\xi')\Big)A(\xi',\xi)\\
&= i\int d\xi\ \theta(\xi)\int d\xi'\ {\rm tr}\Big(\sigma^3(\rho_hC_\Delta)(\xi-\xi')\,A(\xi',\xi)-A(\xi,\xi')\,(\rho_hC_\Delta)(\xi'-\xi)\sigma^3\Big)
\end{aligned}$$

Since the derived Ward identity is valid for all smooth functions $\theta(\xi)$ we have

$$\begin{aligned}
0=&-i\big\langle\bar{\mathbf{\Psi}}(\xi)\sigma^3\mathbf{\Phi}(\xi)\big\rangle+i\big\langle\bar{\mathbf{\Phi}}(\xi)\sigma^3\mathbf{\Psi}(\xi)\big\rangle+2\big\langle\bar{\mathbf{\Psi}}(\xi)J^\#(\xi)\mathbf{\Psi}(\xi)\big\rangle-2(1-r)\big\langle\bar{\mathbf{\Psi}}(\xi)\Delta^\#\mathbf{\Psi}(\xi)\big\rangle\\
&+i\int d\xi'\ {\rm tr}\Big(\sigma^3(\rho_hC_\Delta)(\xi-\xi')\,A(\xi',\xi)-A(\xi,\xi')\,(\rho_hC_\Delta)(\xi'-\xi)\sigma^3\Big)\\
&+R\\
=&-i\big\langle\bar{\mathbf{\Psi}}(\xi)\sigma^3\mathbf{\Phi}(\xi)\big\rangle+i\big\langle\bar{\mathbf{\Phi}}(\xi)\sigma^3\mathbf{\Psi}(\xi)\big\rangle+2\big\langle\bar{\mathbf{\Psi}}(\xi)J^\#(\xi)\mathbf{\Psi}(\xi)\big\rangle-2(1-r)\big\langle\bar{\mathbf{\Psi}}(\xi)\Delta^\#\mathbf{\Psi}(\xi)\big\rangle\\
&+i\ {\rm tr}\Big(\sigma^3(\rho_hC_\Delta A)(\xi,\xi)-(A\rho_hC_\Delta)(\xi,\xi)\sigma^3\Big)\\
&+R
\end{aligned}$$

where R is a remainder that is independent of J, Φ and $\bar\Phi$.

To identify the cancellation between $2\left\langle \bar{\mathbf{\Psi}}(\xi)\Delta^{\#}\mathbf{\Psi}(\xi)\right\rangle$ and the terms containing A, we use the integration by parts formulae

$$\int \mathbf{\Psi}(\xi)\, U(\mathbf{\Psi},\bar{\mathbf{\Psi}})\, d\mu_P(\mathbf{\Psi},\bar{\mathbf{\Psi}}) = \int d\mu_P(\mathbf{\Psi},\bar{\mathbf{\Psi}}) \int d\xi'\, P(\xi,\xi')\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\xi')}U(\mathbf{\Psi},\bar{\mathbf{\Psi}})$$
$$\int U(\mathbf{\Psi},\bar{\mathbf{\Psi}})\, \bar{\mathbf{\Psi}}(\xi')\, d\mu_P(\mathbf{\Psi},\bar{\mathbf{\Psi}}) = \int d\mu_P(\mathbf{\Psi},\bar{\mathbf{\Psi}}) \int d\xi\, \Big(U(\mathbf{\Psi},\bar{\mathbf{\Psi}})\tfrac{\delta}{\delta\mathbf{\Psi}(\xi)}\Big)\, P(\xi,\xi')$$

with $P=\rho_h C_\Delta$ and the notation $\mathcal{N}=\int e^{\mathcal{A}_R^{\rm int}}\, d\mu_P$. For any two by two matrix M

$$\begin{aligned}
\mathcal{N}\left\langle \bar{\mathbf{\Psi}}(\xi_1)M\mathbf{\Psi}(\xi_2)\right\rangle &= \int \bar{\mathbf{\Psi}}(\xi_1)M\mathbf{\Psi}(\xi_2)e^{\mathcal{A}_R^{\rm int}}\ d\mu_P = -\int {\rm tr}\, M\mathbf{\Psi}(\xi_2)\bar{\mathbf{\Psi}}(\xi_1)\ e^{\mathcal{A}_R^{\rm int}}\ d\mu_P \\
&= -\int d\mu_P \int d\eta\, {\rm tr}\, M\left(\mathbf{\Psi}(\xi_2)e^{\mathcal{A}_R^{\rm int}}\right)\tfrac{\delta}{\delta\mathbf{\Psi}(\eta)}P(\eta,\xi_1) \\
&= -\,{\rm tr}\ P(\xi_2,\xi_1)M\mathcal{N} - \int d\mu_P \int d\eta\ {\rm tr}\, M\ \mathbf{\Psi}(\xi_2)\left(e^{\mathcal{A}_R^{\rm int}}\tfrac{\delta}{\delta\mathbf{\Psi}(\eta)}\right)P(\eta,\xi_1) \\
&= -\,{\rm tr}\ P(\xi_2,\xi_1)M\mathcal{N} - \int d\mu_P \int d\eta d\eta'\ {\rm tr}\, MP(\xi_2,\eta')\tfrac{\delta}{\delta\bar{\mathbf{\Psi}}(\eta')}e^{\mathcal{A}_R^{\rm int}}\tfrac{\delta}{\delta\mathbf{\Psi}(\eta)}P(\eta,\xi_1) \\
&= -\,{\rm tr}\ P(\xi_2,\xi_1)M\mathcal{N} - \mathcal{N}\int d\eta d\eta'\ {\rm tr}\, MP(\xi_2,\eta')A(\eta',\eta)P(\eta,\xi_1)
\end{aligned}$$

or, dividing by $\mathcal{N}$,

$$\begin{aligned}
\left\langle \bar{\mathbf{\Psi}}(\xi_1)M\mathbf{\Psi}(\xi_2)\right\rangle &= -\,{\rm tr}\ P(\xi_2,\xi_1)M - \int d\eta d\eta'\, {\rm tr}\ M(\rho_hC_\Delta)(\xi_2-\eta')A(\eta',\eta)(\rho_hC_\Delta)(\eta-\xi_1) \\
&= -\,{\rm tr}\ M\big(\rho_hC_\Delta\big)(\xi_2-\xi_1) - {\rm tr}\ M(\rho_hC_\Delta A\rho_hC_\Delta)(\xi_2,\xi_1)
\end{aligned}$$

In particular

$$\left\langle \bar{\mathbf{\Psi}}(\xi)\Delta^{\#}\mathbf{\Psi}(\xi)\right\rangle = -\,{\rm tr}\ \Delta^{\#}\big(\rho_hC_\Delta\big)(\xi=0) - {\rm tr}\ \Delta^{\#}(\rho_hC_\Delta A\rho_hC_\Delta)(\xi,\xi)$$

Substituting, the derived Ward identity becomes

$$\begin{aligned}
&2r\left\langle \bar{\mathbf{\Psi}}(\xi)\Delta^{\#}\mathbf{\Psi}(\xi)\right\rangle \\
=&i\left\langle \bar{\mathbf{\Psi}}(\xi)\sigma^3\mathbf{\Phi}(\xi)\right\rangle - i\left\langle \bar{\mathbf{\Phi}}(\xi)\sigma^3\mathbf{\Psi}(\xi)\right\rangle - 2\left\langle \bar{\mathbf{\Psi}}(\xi)J^{\#}(\xi)\mathbf{\Psi}(\xi)\right\rangle + 2\left\langle \bar{\mathbf{\Psi}}(\xi)\Delta^{\#}\mathbf{\Psi}(\xi)\right\rangle \\
&-i\ {\rm tr}\left(\sigma^3(\rho_hC_\Delta A)(\xi,\xi) - (A\rho_hC_\Delta)(\xi,\xi)\sigma^3\right) - R \\
=&i\left\langle \bar{\mathbf{\Psi}}(\xi)\sigma^3\mathbf{\Phi}(\xi)\right\rangle - i\left\langle \bar{\mathbf{\Phi}}(\xi)\sigma^3\mathbf{\Psi}(\xi)\right\rangle - 2\left\langle \bar{\mathbf{\Psi}}(\xi)J^{\#}(\xi)\mathbf{\Psi}(\xi)\right\rangle \\
&-i\ {\rm tr}\left(\sigma^3(\rho_hC_\Delta A)(\xi,\xi) - (A\rho_hC_\Delta)(\xi,\xi)\sigma^3 - 2i\,\Delta^{\#}(\rho_hC_\Delta A\rho_hC_\Delta)(\xi,\xi)\right) \\
&-R'
\end{aligned}$$

where now $2{\rm tr}\ \big(\rho_hC_\Delta\big)(\xi{=}0)\Delta^{\#}$ appears in the $J,\mathbf{\Phi},\bar{\mathbf{\Phi}}$-independent remainder R'.

Multiplying and dividing by C_Δ

$$
\begin{aligned}
&-i\ \mathrm{tr}\Big(\sigma^3\rho_h C_\Delta A - A\rho_h C_\Delta\sigma^3 - 2i\,\Delta^{\#}\rho_h C_\Delta A\rho_h C_\Delta\Big)\\
&= -i\ \mathrm{tr}\Big(\sigma^3\rho_h C_\Delta A C_\Delta(ik_0 e(\mathbf{k})\sigma^3\Delta) - (ik_0 e(\mathbf{k})\sigma^3\Delta)C_\Delta A C_\Delta\rho_h\sigma^3\\
&\qquad\qquad - 2i\,\Delta^{\#}\rho_h C_\Delta A C_\Delta\rho_h\Big)\\
&= -i\ \mathrm{tr}\Big(\sigma^3\rho_h C_\Delta A C_\Delta(ik_0 e(\mathbf{k})\sigma^3) - (ik_0 e(\mathbf{k})\sigma^3)C_\Delta A C_\Delta\rho_h\sigma^3\Big)\\
&\quad + \mathrm{tr}\Big((i\Delta\sigma^3 - i\sigma^3\Delta - 2\Delta^{\#})\rho_h C_\Delta A C_\Delta\rho_h\Big)\\
&\quad - i\ \mathrm{tr}\Big(-\Delta\sigma^3\rho_h C_\Delta A C_\Delta(1-\rho_h) + (1-\rho_h)\sigma^3\Delta C_\Delta A C_\Delta\rho_h\Big)\\
&= -i\ \mathrm{tr}\Big(\sigma^3\rho_h C_\Delta A C_\Delta(ik_0 e(\mathbf{k})\sigma^3) - (ik_0 e(\mathbf{k})\sigma^3)C_\Delta A C_\Delta\rho_h\sigma^3\Big)\\
&\quad + \mathrm{tr}\Big(\Delta^{\#}\rho_h C_\Delta A C_\Delta(1-\rho_h) + (1-\rho_h)\Delta^{\sharp}C_\Delta A C_\Delta\rho_h\Big)\\
&= -i\ \mathrm{tr}\Big(\sigma^3\rho_h C_\Delta A\rho_h C_\Delta(ik_0 e(\mathbf{k})\sigma^3) - \sigma^3(ik_0 e(\mathbf{k})\sigma^3)\rho_h C_\Delta A C_\Delta\rho_h\Big)\\
&\quad + \mathrm{tr}\Big(\Delta^{\#}\rho_h C_\Delta A C_\Delta(1-\rho_h) + (1-\rho_h)\Delta^{\sharp}C_\Delta A C_\Delta\rho_h\Big)\\
&\quad - i\ \mathrm{tr}\Big(\sigma^3\rho_h C_\Delta A(1-\rho_h)C_\Delta(ik_0 e(\mathbf{k})\sigma^3) - \sigma^3(ik_0 e(\mathbf{k})\sigma^3)(1-\rho_h)C_\Delta A C_\Delta\rho_h\Big)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&2r\left\langle\bar{\mathbf{\Psi}}(\xi)\Delta^{\#}\mathbf{\Psi}(\xi)\right\rangle\\
&= i\left\langle\bar{\mathbf{\Psi}}(\xi)\sigma^3\mathbf{\Phi}(\xi)\right\rangle - i\left\langle\bar{\mathbf{\Phi}}(\xi)\sigma^3\mathbf{\Psi}(\xi)\right\rangle - 2\left\langle\bar{\mathbf{\Psi}}(\xi)J^{\#}(\xi)\mathbf{\Psi}(\xi)\right\rangle\\
&\quad - i\ \mathrm{tr}\Big(\sigma^3\rho_h C_\Delta A\rho_h C_\Delta(ik_0 e(\mathbf{k})\sigma^3) - \sigma^3(ik_0 e(\mathbf{k})\sigma^3)\rho_h C_\Delta A\rho_h C_\Delta\Big)(\xi,\xi)\\
&\quad - R' + U
\end{aligned}
$$

where the remainder

$$
\begin{aligned}
U &= \mathrm{tr}\Big(\Delta^{\#}\rho_h C_\Delta A C_\Delta(1-\rho_h) + (1-\rho_h)\Delta^{\sharp}C_\Delta A C_\Delta\rho_h\Big)(\xi,\xi)\\
&\quad - i\,\mathrm{tr}\Big(\sigma^3\rho_h C_\Delta A(1-\rho_h)C_\Delta(ik_0 e(\mathbf{k})\sigma^3) - \sigma^3(ik_0 e(\mathbf{k})\sigma^3)(1-\rho_h)C_\Delta A C_\Delta\rho_h\Big)(\xi,\xi)
\end{aligned}
$$

can be expected to disappear when all cutoffs are removed.

If $I(\xi,\eta)$ is the kernel of an integral operator, then the kernel of the commutator

$$
(ik_0 I - Iik_0)(\xi,\eta) = -\left(\frac{\partial}{\partial\xi_0} + \frac{\partial}{\partial\eta_0}\right)I(\xi,\eta)
$$

since

$$
\begin{aligned}
(ik_0 If)(\xi) &= -\int d\eta\ \frac{\partial}{\partial\xi_0}I(\xi,\eta)f(\eta)\\
(Iik_0 f)(\xi) &= -\int d\eta\ I(\xi,\eta)\frac{\partial}{\partial\eta_0}f(\eta)\\
&= \int d\eta\ \left(\frac{\partial}{\partial\eta_0}I(\xi,\eta)\right)f(\eta)
\end{aligned}
$$

So, by the chain rule

$$(ik_0 I - I ik_0)(\xi,\xi) \;=\; -\frac{\partial}{\partial \xi_0} I(\xi,\xi)$$

Consequently,

$$\begin{aligned}
&-i\ \mathrm{tr}\left(\sigma^3\rho_h C_\Delta A\rho_h C_\Delta(ik_0 e(\mathbf{k})\sigma^3) - \sigma^3(ik_0 e(\mathbf{k})\sigma^3)\rho_h C_\Delta A\rho_h C_\Delta\right)(\xi,\xi)\\
&= -i\frac{\partial}{\partial \xi_0}\mathrm{tr}\left(\sigma^3\rho_h C_\Delta A\rho_h C_\Delta\right)(\xi,\xi) + \tfrac{i}{2m}\mathrm{tr}\left(\rho_h C_\Delta A\rho_h C_\Delta \mathbf{k}^2 - \mathbf{k}^2\rho_h C_\Delta A\rho_h C_\Delta\right)(\xi,\xi)\\
&= -i\frac{\partial}{\partial \xi_0}\mathrm{tr}\left(\sigma^3\rho_h C_\Delta A\rho_h C_\Delta\right)(\xi,\xi) + \tfrac{i}{2m}\sum_{j=1}^{d}\mathrm{tr}\left(\rho_h C_\Delta A\rho_h C_\Delta \mathbf{k}_j^2 - \mathbf{k}_j\rho_h C_\Delta A\rho_h C_\Delta\mathbf{k}_j\right)(\xi,\xi)\\
&\quad + \tfrac{i}{2m}\sum_{j=1}^{d}\mathrm{tr}\left(\mathbf{k}_j\rho_h C_\Delta A\rho_h C_\Delta \mathbf{k}_j - \mathbf{k}_j^2\rho_h C_\Delta A\rho_h C_\Delta\right)(\xi,\xi)\\
&= -i\frac{\partial}{\partial \xi_0}\mathrm{tr}\left(\sigma^3\rho_h C_\Delta A\rho_h C_\Delta\right)(\xi,\xi) - \tfrac{1}{2m}\sum_{j=1}^{d}\frac{\partial}{\partial \xi_j}\mathrm{tr}\left(\rho_h C_\Delta A\rho_h C_\Delta \mathbf{k}_j\right)(\xi,\xi)\\
&\quad - \tfrac{1}{2m}\sum_{j=1}^{d}\frac{\partial}{\partial \xi_j}\mathrm{tr}\left(\mathbf{k}_j\rho_h C_\Delta A\rho_h C_\Delta\right)(\xi,\xi)
\end{aligned}$$

Extracting another derivative

$$\begin{aligned}
&-i\ \mathrm{tr}\left(\sigma^3\rho_h C_\Delta A\rho_h C_\Delta(ik_0 e(\mathbf{k})\sigma^3) - \sigma^3(ik_0 e(\mathbf{k})\sigma^3)\rho_h C_\Delta A\rho_h C_\Delta\right)(\xi,\xi)\\
&= -i\frac{\partial}{\partial \xi_0}\mathrm{tr}\left(\sigma^3\rho_h C_\Delta A\rho_h C_\Delta\right)(\xi,\xi) - \tfrac{1}{2m}\sum_{j=1}^{d}\frac{\partial}{\partial \xi_j}\left[i\frac{\partial}{\partial \eta_j}\mathrm{tr}\left(\rho_h C_\Delta A\rho_h C_\Delta\right)(\xi,\eta)\big|_{\eta=\xi}\right]\\
&\quad - \tfrac{1}{2m}\sum_{j=1}^{d}\frac{\partial}{\partial \xi_j}\left[-i\frac{\partial}{\partial \xi_j}\mathrm{tr}\left(\rho_h C_\Delta A\rho_h C_\Delta\right)(\xi,\eta)\big|_{\eta=\xi}\right]\\
&= i\partial_0\left\langle\bar{\mathbf{\Psi}}(\xi)\sigma^3\mathbf{\Psi}(\xi)\right\rangle - \tfrac{i}{2m}\sum_{j=1}^{d}\partial_j\left\langle(\partial_j\bar{\mathbf{\Psi}})(\xi)\mathbf{\Psi}(\xi)\right\rangle + \tfrac{i}{2m}\sum_{j=1}^{d}\partial_j\left\langle\bar{\mathbf{\Psi}}(\xi)(\partial_j\mathbf{\Psi})(\xi)\right\rangle\\
&\quad + \text{terms independent of } \mathbf{\Phi},\bar{\mathbf{\Phi}}
\end{aligned}$$

Recall that

$$\left\langle F(\mathbf{\Psi},\bar{\mathbf{\Psi}})\right\rangle = \frac{\int F(\mathbf{\Psi},\bar{\mathbf{\Psi}})e^{\mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,P)}\ \prod_{k,i}d\mathbf{\Psi}^i(k)\,d\bar{\mathbf{\Psi}}_i(k)}{\int e^{\mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}},\mathbf{\Phi},\bar{\mathbf{\Phi}},J;r,P)}\ \prod_{k,i}d\mathbf{\Psi}^i(k)\,d\bar{\mathbf{\Psi}}_i(k)}\ .$$

depends on the external fields J, $\mathbf{\Phi}$ and $\bar{\mathbf{\Phi}}$.

Finally, the derived Ward identity becomes

$$\begin{aligned}
&2r\left\langle\bar{\mathbf{\Psi}}(\xi)\Delta^{\#}\mathbf{\Psi}(\xi)\right\rangle - i\partial_0\left\langle\bar{\mathbf{\Psi}}(\xi)\sigma^3\mathbf{\Psi}(\xi)\right\rangle + \tfrac{i}{2m}\sum_{j=1}^{d}\partial_j\left\langle(\partial_j\bar{\mathbf{\Psi}})(\xi)\mathbf{\Psi}(\xi) - \bar{\mathbf{\Psi}}(\xi)(\partial_j\mathbf{\Psi})(\xi)\right\rangle\\
&=\; i\left\langle\bar{\mathbf{\Psi}}(\xi)\sigma^3\mathbf{\Phi}(\xi)\right\rangle - i\left\langle\bar{\mathbf{\Phi}}(\xi)\sigma^3\mathbf{\Psi}(\xi)\right\rangle - 2\left\langle\bar{\mathbf{\Psi}}(\xi)J^{\#}(\xi)\mathbf{\Psi}(\xi)\right\rangle - R'' + U
\end{aligned}$$

as claimed.

Now apply $\int d\xi e^{-i<q,\xi>_-}$ to both sides of the derived Ward identity.

$$
\begin{aligned}
&2r\Big\langle \int đk\ \bar{\mathbf{\Psi}}(k+q)\Delta^{\#}\mathbf{\Psi}(k)\Big\rangle - q_0\Big\langle \int đk\ \bar{\mathbf{\Psi}}(k+q)\sigma^3\mathbf{\Psi}(k)\Big\rangle \\
&+\tfrac{i}{2m}\sum_{j=1}^{d} q_j \int đk\Big\langle (q_j+k_j)\bar{\mathbf{\Psi}}(k+q)\mathbf{\Psi}(k) + k_j\bar{\mathbf{\Psi}}(k+q)\mathbf{\Psi}(k)\Big\rangle \\
&\qquad = i\Big\langle \int đk\ \bar{\mathbf{\Psi}}(k+q)\sigma^3\mathbf{\Phi}(k)\Big\rangle - i\Big\langle \int đk\ \bar{\mathbf{\Phi}}(k+q)\sigma^3\mathbf{\Psi}(k)\Big\rangle \\
&\qquad\quad - 2\Big\langle \int đk\, đp\ \bar{\mathbf{\Psi}}(k+p+q)\widetilde{J}(p)^{\#}\mathbf{\Psi}(k)\Big\rangle - \widetilde{R}'' + \widetilde{U}
\end{aligned}
$$

or

$$
\begin{aligned}
&\Big\langle \int đk\ \bar{\mathbf{\Psi}}(k+q)\Big(2r\Delta^{\#} - q_0\sigma^3 + \tfrac{i}{2m}\mathbf{q}^2 + \tfrac{i}{m}\langle \mathbf{q},\mathbf{k}\rangle\Big)\mathbf{\Psi}(k)\Big\rangle \\
&\ = \ i\Big\langle \int đk\ \Big(\bar{\mathbf{\Psi}}(k+q)\sigma^3\mathbf{\Phi}(k) - \bar{\mathbf{\Phi}}(k+q)\sigma^3\mathbf{\Psi}(k)\Big)\Big\rangle - 2\Big\langle \int đk\, đp\ \bar{\mathbf{\Psi}}(k+p+q)\widetilde{J}(p)^{\#}\mathbf{\Psi}(k)\Big\rangle \\
&\qquad - \widetilde{R}'' + \widetilde{U}
\end{aligned}
$$

■

Let

$$
\big\langle F(\mathbf{\Psi},\bar{\mathbf{\Psi}})\big\rangle_0 = \frac{\int F(\mathbf{\Psi},\bar{\mathbf{\Psi}})e^{\mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}},0,0,0;r,P)}\ \prod_{k,i} d\mathbf{\Psi}^i(k)\, d\bar{\mathbf{\Psi}}_i(k)}{\int e^{\mathcal{A}_R(\mathbf{\Psi},\bar{\mathbf{\Psi}},0,0,0;r,P)}\ \prod_{k,i} d\mathbf{\Psi}^i(k)\, d\bar{\mathbf{\Psi}}_i(k)}
$$

denote the expectation with the external fields $\mathbf{\Phi},\bar{\mathbf{\Phi}}$ and J set to zero. Recall that the two point function is

$$
\Big(\big\langle \mathbf{\Psi}^i(k)\bar{\mathbf{\Psi}}_j(p)\big\rangle\Big) = (2\pi)^{d+1}\delta(k-p)S(k)
$$

Proposition IV.3 *Suppose that the interaction*

$$
\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle \ = \ \widehat{V}(k_1k_3)
$$

on the support of the delta function $\delta(k_1+k_2-k_3-k_4)$. *If the BCS constraints are satisfied, then*

$$
\begin{aligned}
&\Big\langle \mathbf{\Psi}(t)\bar{\mathbf{\Psi}}(t+q+p)\ ;\int đk\ \bar{\mathbf{\Psi}}(k+q)\Big(2r\Delta^{\#} - q_0\sigma^3 + \tfrac{i}{2m}\mathbf{q}^2 + \tfrac{i}{m}\langle\mathbf{q},\mathbf{k}\rangle\Big)\mathbf{\Psi}(k)\Big\rangle_0 \\
&\ = \ i\Big(S(t)\sigma^3 - \sigma^3 S(t+\mathbf{q})\Big)(2\pi)^{d+1}\delta(p) + \tfrac{\delta}{\delta\bar{\mathbf{\Phi}}(t)}\widetilde{U}\tfrac{\delta}{\delta\mathbf{\Phi}(t+q+p)}(0)
\end{aligned}
$$

Proof: Recall that the expectation depends on the external fields. Differentiating on the left with respect to $\bar{\mathbf{\Phi}}(p_1)$ and on the right with respect to $\mathbf{\Phi}(p_2)$ and setting $\mathbf{\Phi} = \bar{\mathbf{\Phi}} = 0$ yields

$$\begin{aligned}
&\Big\langle \mathbf{\Psi}(p_1)\bar{\mathbf{\Psi}}(p_2)\ ;\int đk\ \bar{\mathbf{\Psi}}(k{+}q)\Big(2r\Delta^{\#} - q_0\sigma^3 + \tfrac{i}{2m}\mathbf{q}^2 + \tfrac{i}{m}\langle \mathbf{q}, \mathbf{k}\rangle\Big)\mathbf{\Psi}(k)\Big\rangle_0 \\
&= i\Big\langle \mathbf{\Psi}(p_1)\bar{\mathbf{\Psi}}(p_2{+}q)\sigma^3 - \sigma^3\mathbf{\Psi}(p_1 - q)\bar{\mathbf{\Psi}}(p_2)\Big\rangle_0 + \tfrac{\delta}{\delta\bar{\mathbf{\Phi}}(p_1)}\widetilde{U}\tfrac{\delta}{\delta\mathbf{\Phi}(p_2)}(0) \\
&= i\Big(S(p_1)\sigma^3 - \sigma^3 S(p_2)\Big)(2\pi)^{d+1}\delta(p_1p_2q) + \tfrac{\delta}{\delta\bar{\mathbf{\Phi}}(p_1)}\widetilde{U}\tfrac{\delta}{\delta\mathbf{\Phi}(p_2)}(0)
\end{aligned}$$

Finally make the change of variables

$$\begin{aligned} p_1 &= t \\ p_2 &= t + q + p \end{aligned}$$

■

Introduce the notation

$$\begin{aligned}
\mathcal{Q}_q(A(\xi)) &= \int d\xi\ e^{-i<q,\xi>-}\,\bar{\mathbf{\Psi}}(\xi)A(\xi)\mathbf{\Psi}(\xi) \\
&= \int đk\ đp\ \bar{\mathbf{\Psi}}(k{+}p{+}q)A(p)\mathbf{\Psi}(k)
\end{aligned}$$

for a general quadratic polynomial in the Grassmann algebra. When $A(\xi)$ is a constant in position space, its Fourier transform has a delta function forcing $p = 0$ and $\mathcal{Q}$ simplifies to

$$\mathcal{Q}_q(A) = \int đk\ \bar{\mathbf{\Psi}}(k{+}q)A\mathbf{\Psi}(k)$$

Define

$$\begin{aligned}
&\big\langle \mathcal{Q}_{p_1}(\Delta);\cdots;\mathcal{Q}_{p_n}(\Delta);\mathcal{Q}_{q_1}(\Delta^{\#});\cdots;\mathcal{Q}_{q_m}(\Delta^{\#})\big\rangle_0 \\
&= (2\pi)^{(d+1)(m+n)}\tfrac{\delta}{\delta j(p_1)}\cdots\tfrac{\delta}{\delta j(p_n)}\tfrac{\delta}{\delta j^{\#}(q_1)}\cdots\tfrac{\delta}{\delta j^{\#}(q_m)}\mathcal{S}_R(0,0,j\Delta + j^{\#}\Delta^{\#};r,\rho_h C_\Delta)\big|_{j=j^{\#}=0}
\end{aligned}$$

Proposition IV.4 *Suppose that the interaction*

$$\langle k_1, k_2|\mathrm{V}|k_3, k_4\rangle \;=\; \widehat{V}(k_1k_3)$$

on the support of the delta function $\delta(k_1{+}k_2{-}k_3{-}k_4)$. *If the BCS constraints are satisfied, then*

$$\begin{aligned}
&\big\langle \mathcal{Q}_q\left(2r\Delta^{\#} - q_0\sigma^3 + \tfrac{i}{2m}\mathbf{q}^2 + \tfrac{i}{m}\langle \mathbf{q},\mathbf{k}\rangle\right);\mathcal{Q}_{p_1}(\Delta);\cdots;\mathcal{Q}_{p_n}(\Delta);\mathcal{Q}_{q_1}(\Delta^{\#});\cdots;\mathcal{Q}_{q_m}(\Delta^{\#})\big\rangle_0 \\
&= -\,2\sum_{i=1}^{n}\big\langle \mathcal{Q}_{q+p_i}(\Delta^{\#});\mathcal{Q}_{p_1}(\Delta);\cdots;\mathcal{Q}_{\not{p}_i}(\Delta);\cdots;\mathcal{Q}_{q_m}(\Delta^{\#})\big\rangle_0 \\
&\quad + 2\sum_{i=1}^{m}\big\langle \mathcal{Q}_{q+q_i}(\Delta);\mathcal{Q}_{p_1}(\Delta);\cdots;\mathcal{Q}_{\not{q}_i}(\Delta^{\#});\cdots;\mathcal{Q}_{q_m}(\Delta^{\#})\big\rangle_0 \\
&\quad + (2\pi)^{(d+1)(m+n)}\tfrac{\delta}{\delta j(p_1)}\cdots\tfrac{\delta}{\delta j(p_n)}\tfrac{\delta}{\delta j^{\#}(q_1)}\cdots\tfrac{\delta}{\delta j^{\#}(q_m)}\widetilde{U}\big|_{j=j^{\#}=0}
\end{aligned}$$

Proof: Upon evaluation at $\Phi = \bar\Phi = 0$ and $J = j\Delta + j^\#\Delta^\#$ the Ward identity of Proposition IV.2 becomes

$$\left\langle \mathcal{Q}_q\left(2r\Delta^\# - q_0\sigma^3 + \tfrac{i}{2m}\mathbf{q}^2 + \tfrac{i}{m}\left\langle \mathbf{q},\mathbf{k}\right\rangle\right)\right\rangle = -2\left\langle \mathcal{Q}_q(j\Delta^\# - j^\#\Delta)\right\rangle + \widetilde{U} + \text{ terms indep of } J$$

Now apply $(2\pi)^{(d+1)(m+n)}\frac{\delta}{\delta j(p_1)}\cdots\frac{\delta}{\delta j(p_n)}\frac{\delta}{\delta j^\#(q_1)}\cdots\frac{\delta}{\delta j^\#(q_m)}$ and set $j = j^\# = 0$ to give

$$\begin{aligned}
&\left\langle \mathcal{Q}_q\left(2r\Delta^\# - q_0\sigma^3 + \tfrac{i}{2m}\mathbf{q}^2 + \tfrac{i}{m}\left\langle \mathbf{q},\mathbf{k}\right\rangle\right); \mathcal{Q}_{p_1}(\Delta);\cdots;\mathcal{Q}_{p_n}(\Delta);\mathcal{Q}_{q_1}(\Delta^\#);\cdots;\mathcal{Q}_{q_m}(\Delta^\#)\right\rangle_0\\
&\quad = -2\sum_{i=1}^{n}\left\langle \mathcal{Q}_{q+p_i}(\Delta^\#);\mathcal{Q}_{p_1}(\Delta);\cdots;\mathcal{Q}_{\not p_i}(\Delta);\cdots;\mathcal{Q}_{q_m}(\Delta^\#)\right\rangle_0\\
&\qquad + 2\sum_{i=1}^{m}\left\langle \mathcal{Q}_{q+q_i}(\Delta);\mathcal{Q}_{p_1}(\Delta);\cdots;\mathcal{Q}_{\not q_i}(\Delta^\#);\cdots;\mathcal{Q}_{q_m}(\Delta^\#)\right\rangle_0\\
&\qquad + (2\pi)^{(d+1)(m+n)}\frac{\delta}{\delta j(p_1)}\cdots\frac{\delta}{\delta j(p_n)}\frac{\delta}{\delta j^\#(q_1)}\cdots\frac{\delta}{\delta j^\#(q_m)}\widetilde{U}\Big|_{j=j^\#=0}
\end{aligned}$$

since, for example,

$$\begin{aligned}
(2\pi)^{d+1}\frac{\delta}{\delta j(p_1)}\mathcal{Q}_q(j\Delta^\# - j^\#\Delta) &= (2\pi)^{d+1}\frac{\delta}{\delta j(p_1)}\int đk\, đp\ \bar{\mathbf{\Psi}}(k{+}p{+}q)(j(p)\Delta^\# - j^\#(p)\Delta)\mathbf{\Psi}(k)\\
&= \int đk\ \bar{\mathbf{\Psi}}(k{+}p_1{+}q)\Delta^\#\mathbf{\Psi}(k)\\
&= \mathcal{Q}_{q+p_1}(\Delta^\#)
\end{aligned}$$

■

Corollary IV.5 *Under the hypotheses of Proposition IV.4*

$$\begin{aligned}
&r\left\langle \mathcal{Q}_0(\Delta^\#);\mathcal{Q}_{p_1}(\Delta);\cdots;\mathcal{Q}_{p_n}(\Delta);\mathcal{Q}_{q_1}(\Delta^\#);\cdots;\mathcal{Q}_{q_m}(\Delta^\#)\right\rangle_0\\
&\quad = -\sum_{i=1}^{n}\left\langle \mathcal{Q}_{p_i}(\Delta^\#);\mathcal{Q}_{p_1}(\Delta);\cdots;\mathcal{Q}_{\not p_i}(\Delta);\cdots;\mathcal{Q}_{q_m}(\Delta^\#)\right\rangle_0\\
&\qquad + \sum_{i=1}^{m}\left\langle \mathcal{Q}_{q_i}(\Delta);\mathcal{Q}_{p_1}(\Delta);\cdots;\mathcal{Q}_{\not q_i}(\Delta^\#);\cdots;\mathcal{Q}_{q_m}(\Delta^\#)\right\rangle_0\\
&\qquad + \tfrac{1}{2}(2\pi)^{(d+1)(m+n)}\frac{\delta}{\delta j(p_1)}\cdots\frac{\delta}{\delta j(p_n)}\frac{\delta}{\delta j^\#(q_1)}\cdots\frac{\delta}{\delta j^\#(q_m)}\widetilde{U}\Big|_{j=j^\#=0}
\end{aligned}$$

In particular, when all the external momenta are zero

$$\begin{aligned}
r\left\langle (\mathcal{Q}_0(\Delta);)^n\left(\mathcal{Q}_0(\Delta^\#);\right)^{m+1}\right\rangle_0 &= -\,n\left\langle (\mathcal{Q}_0(\Delta);)^{n-1}\left(\mathcal{Q}_0(\Delta^\#);\right)^{m+1}\right\rangle_0\\
&\quad + m\left\langle (\mathcal{Q}_0(\Delta);)^{n+1}\left(\mathcal{Q}_0(\Delta^\#);\right)^{m-1}\right\rangle_0\\
&\quad + \tfrac{1}{2}(2\pi)^{(d+1)(m+n)}\left(\frac{\delta}{\delta j(0)}\right)^n\left(\frac{\delta}{\delta j^\#(0)}\right)^m\widetilde{U}\Big|_{j=j^\#=0}
\end{aligned}$$

§V Power Counting and the Goldstone Boson Propagator

In this section we write the model (I.1) in terms of an intermediate boson field and examine its power counting. Superficially, the power counting is nonrenormalizable. However, Corollary IV.5 is applied to show that this is not the case. We also use the results of §III to define the Goldstone Boson propagator in perturbation theory.

We split the interaction (II.3) into an "effective interaction" and an "irrelevant" remainder. In [2, 6] it is shown that the dominant part of the interaction $\left\langle t+\frac{q}{2},-t+\frac{q}{2}\middle|V\middle|s+\frac{q}{2},-s+\frac{q}{2}\right\rangle$ is that with $t\approx t'$, $s\approx s'$ and $q\approx 0$. Recall that $t'=(0,k_F\mathbf{t}/|\mathbf{t}|)$ is the projection of t on the Fermi surface. So, consider the reduced interaction

$$\frac{\lambda}{2}\int \,đs\; đt\; đq\;\; \bar\psi_{(t+\frac{q}{2})}\,\psi_{(s+\frac{q}{2})}\;\langle t',-t'|V|s',-s'\rangle\;\bar\psi_{(-t+\frac{q}{2})}\,\psi_{(-s+\frac{q}{2})}$$

In this paper we have assumed that the number symmetry is broken in the zero angular momentum sector. In other words, the zero angular momentum coupling constant λ_0 of the decomposition $\lambda\,\langle t',-t'|V|s',-s'\rangle=\sum_{n\ge 0}\lambda_n\pi_n(t',s')$ into spherical harmonics obeys $\lambda_0<0$, $|\lambda_0|\gg|\lambda_j|$ for all $j>0$. We remark that in [6] the coupling constants λ_j were defined by $\lambda\,\langle t',-t'|V|s',-s'\rangle=-\sum_{n\ge 0}\lambda_n\pi_n(t',s')$ making attractive coupling constants positive.

Setting $\lambda_0=-2g^2$, we consider the effective interaction

$$\begin{aligned}\mathcal{V}_{\rm eff} &= -2g^2\int đs\; đt\; đq\;\; \bar\psi_{\uparrow(t+\frac{q}{2})}\,\bar\psi_{\downarrow(-t+\frac{q}{2})}\,\psi_{\downarrow(-s+\frac{q}{2})}\,\psi_{\uparrow(s+\frac{q}{2})}\\ &= -2g^2\int đp\,đq\,\left(\int đt\;\bar\psi_{\uparrow(t+\frac{p}{2})}\,\bar\psi_{\downarrow(-t+\frac{p}{2})}\right)B(p,-q)\left(\int đs\;\psi_{\downarrow(-s+\frac{q}{2})}\,\psi_{\uparrow(s+\frac{q}{2})}\right)\end{aligned}$$

with $B(p,q)=(2\pi)^{d+1}\delta(p+q)$. If (γ_1,γ_2) is a $\mathbb{C}^2$ valued Gaussian variable with the real, even covariance

$$\langle\gamma_i(p)\gamma_j(q)\rangle = \delta_{i,j}\,B(p,q)$$

then

$$e^{-\mathcal{V}_{\rm eff}} = \int\exp\left(g\int d\xi\;\bar{\mathbf{\Psi}}(\xi)\,\gamma(\xi)\,\mathbf{\Psi}(\xi)\right)d\mu(\gamma)$$

where $\gamma=\sigma^1\gamma^1+\sigma^2\gamma^2$.

Performing the fermionic integration

$$\begin{aligned}\int e^{-\mathcal{V}_{\rm eff}}\,d\mu(\mathbf{\Psi},\bar{\mathbf{\Psi}}) &= \int\!\!\int\exp\left(g\int d\xi\;\bar{\mathbf{\Psi}}(\xi)\,\gamma(\xi)\,\mathbf{\Psi}(\xi)\right)d\mu(\gamma)d\mu(\mathbf{\Psi},\bar{\mathbf{\Psi}})\\ &= \int\det\left(1\!\!1-g\,C\gamma\right)d\mu(\gamma)\end{aligned}$$

we obtain (the exponential of) an effective interaction for the intermediate boson field γ. Here, γ is a multiplication operator in position space acting on $\mathbb{C}^2$-valued functions on $\mathbb{R}^{d+1}$ and C is the multiplication operator in momentum space given by

$$C(p) = -\rho(p)\frac{ip_0+e(\mathbf{p})\sigma^3}{p_0^2+e(\mathbf{p})^2}$$

where $\rho(p)$ is the characteristic function of the set $\left\{ p \in \mathbb{R}^{d+1} \;\middle|\; p_0^2 + e(\mathbf{p})^2 < 1 \right\}$. Thus $\rho(p)$ imposes an ultraviolet, but no infrared, cutoff on the Fermions.

Formally,

$$\det(\mathbb{1} - g\,C\gamma)\,d\mu(\gamma) \;=\; e^{-\frac{1}{2}\left(\int d\xi\;\gamma(\xi)^2 - \log\det(\mathbb{1}-g\,C\gamma)\right)} \prod_{\xi\in\mathbb{R}^{d+1}} d\gamma(\xi)$$

The mean field effective potential per unit volume is obtained by evaluating the exponent in the line above at a field γ that is constant in position space and normalizing by the volume. The result $\mathcal{E}(|\gamma|)$ is given by [2, §V]

$$\mathcal{E}(r) \;=\; \tfrac{1}{2}r^2 \;-\; \int đp\;\log\left(1 + g^2 \tfrac{\rho(p)}{p_0^2+e(\mathbf{p})^2}\,r^2\right)$$

where $r = \sqrt{\gamma_1^2+\gamma_2^2}$. The graph of the effective potential is a Bordeaux wine bottle or Mexican hat whose absolute minimum is at $g|\gamma|_* \approx \exp\left\{-\frac{\pi}{mg^2}\right\}$ and has depth approximately $\dfrac{m}{4\pi}(g|\gamma|_*)^2$ and curvature at the minimum approximately $\dfrac{m}{\pi}g^2$. The depth of the full, unnormalized, effective potential is enormous due to the volume of space-time.

Symmetry breaking forces the value of γ to be concentrated near some point on the circle $|\gamma| = |\gamma|_*$. The phase is determined by a boundary condition. We may suppose that γ is concentrated near Δ/g. Then it is natural to shift γ by Δ/g and define the radial and tangential components

$$\begin{aligned}\gamma \;&=\; \tfrac{1}{2\Delta^2}\mathrm{tr}\,(\gamma\Delta)\Delta + \tfrac{1}{2\Delta^2}\mathrm{tr}\,(\gamma\Delta^\#)\Delta^\# \\ &=\; \Delta/g + \gamma_{\rm rad}\Delta/|\Delta| + \gamma_{\rm tan}\Delta^\#/|\Delta|\end{aligned}$$

where $|\Delta| = \sqrt{\Delta^2}$. While $\gamma_{\rm rad}$ and $\gamma_{\rm tan}$ are globally defined they can only be interpreted as radial and tangential components in a small neighbourhood of $\gamma = \Delta/g$. In the new variables the measure

$$\begin{aligned}e^{-\mathcal{V}_{\rm eff}}\,d\mu_{(\mathbf{\Psi},\bar{\mathbf{\Psi}})} \;&=\; \int e^{g\int d\xi\;\bar{\mathbf{\Psi}}(\xi)\,\gamma(\xi)\,\mathbf{\Psi}(\xi)} d\mu_{(\gamma)} d\mu_{(\mathbf{\Psi},\bar{\mathbf{\Psi}})} \\ &=\; \text{const} \int e^{g\int d\xi\;\bar{\mathbf{\Psi}}\gamma_s\mathbf{\Psi} + \int d\xi\;\bar{\mathbf{\Psi}}\,\Delta\,\mathbf{\Psi}} e^{-|\Delta|/g\int d\xi\;\gamma_{\rm rad}} d\mu_{(\gamma_s)} d\mu_{(\mathbf{\Psi},\bar{\mathbf{\Psi}})} \\ &=\; \text{const} \int e^{g\int d\xi\;\bar{\mathbf{\Psi}}\gamma_s\mathbf{\Psi}} e^{-|\Delta|/g\int d\xi\;\gamma_{\rm rad}} d\mu_{(\gamma_s)} d\mu_{\Delta\,(\mathbf{\Psi},\bar{\mathbf{\Psi}})}\end{aligned}$$

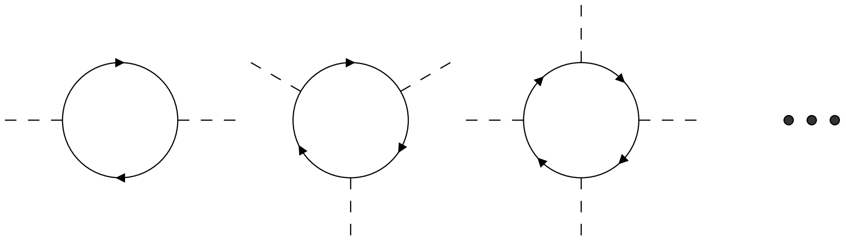
where $\gamma_s = \gamma_{\rm rad}\Delta/|\Delta| + \gamma_{\rm tan}\Delta^\#/|\Delta|$ is the shifted field and $d\mu_\Delta$ is the Grassmann-Gaussian measure with covariance

$$C_\Delta = \frac{\rho(k)}{ik_0 - e(\mathbf{k})\sigma^3 - \Delta\rho(k)}$$

Expanding the integral

$$\int e^{g\int d\xi\;\bar{\mathbf{\Psi}}\gamma_s\mathbf{\Psi}} d\mu_{\Delta\,(\mathbf{\Psi},\bar{\mathbf{\Psi}})} = \det(\mathbb{1} - gC_\Delta\gamma_s)$$

in powers of g generates vertices



The 1-loop cancels $|\Delta|/g \int d\xi\, \gamma_{\mathrm{rad}}$. Absorbing the second order Taylor expansion of the 2-loop around momentum $q = 0$ in the ultralocal meaure $d\mu(\gamma_s)$ yields a Gaussian measure whose covariances for γ_{rad} and γ_{tan} are $\mathrm{const}\, g^{-2}$ and $\mathrm{const}\, \Delta^2 g^{-2}(\mathrm{const}\, q_0^2 + \mathbf{q}^2)^{-1}$.

We now determine the deep infrared power counting of a connected vacuum graph G whose v^{th} generalized vertex is an n_v-loop with $n_v \geq 3$. The restriction $n_v \geq 3$ is justified because the most singular part of the 2-loop has been absorbed in the bosonic measure. The weight of a generalized n_v-loop vertex, arising from its fermion lines, is $g^{n_v}|\Delta|^{2-n_v}$ since the supremum of each of the n_v propagators C_Δ is $1/|\Delta|$ and the single momentum loop integral yields Δ^2. Recall that, in the deep infrared regime the momentum k in the loop satisfies $|k_0|,\ \big||\mathbf{k}| - k_F\big| = O(|\Delta|)$.

To complete the power counting, we introduce a bosonic scale j by multiplying the above propagators by $\chi(M^j < |q| < M^{j+1})$. The following table gives the supremum and the volume of support in momentum space, up to irrelevant constants, of the radial and tangential bosonic propagators.

field	supremum in momentum space	volume of support
γ^j_{rad}	g^{-2}	$M^{(d+1)j}$
γ^j_{tan}	$g^{-2}\Delta^2 M^{-2j}$	$M^{(d+1)j}$

The graph G has $\sum_v n_v/2$ boson lines and $1+\sum_v (n_v/2-1)$ independent bosonic momentum loops. If all the boson lines are γ_{rad} lines, the power counting for G is

$$M^{(d+1)j} \prod_v g^{n_v} |\Delta|^{2-n_v} \left(g^{-2}\right)^{n_v/2} M^{(d+1)j(n_v/2-1)} \ \leq\ M^{(d+1)j} \left(M^{(d+1)j/2}/|\Delta|\right)^{\Sigma_v (n_v-2)}$$

For $d \geq 2$,

$$M^{(d+1)j/2}/|\Delta| \leq M^{3j/2}/|\Delta| \leq M^{j/2}$$

since, in the deep infrared regime $M^j \leq |\Delta|$. In this case we have positive power counting for all generalized vertices.

On the other hand, if all the boson lines are γ_{tan} lines, the power counting for G

is

$$M^{(d+1)j}\prod_v g^{n_v}|\Delta|^{2-n_v}\left(g^{-2}\Delta^2M^{-2j}\right)^{n_v/2}M^{(d+1)j(n_v/2-1)}$$
$$= M^{(d+1)j}\prod_v \Delta^2 M^{\frac{1}{2}(d-1)jn_v-(d+1)j}$$

In each factor v, the coefficient of j in the exponent is $\frac{1}{2}(d-1)n_v-d-1$. When $d=3$, resp. $d=2$, this gives positive power counting for $n_v>4$, resp. $n_v>6$.

To relate the power counting for small n_v to Corollary IV.5 we prove

Proposition V.1 *Let $n\ge 3$. Then*

$$\langle\gamma_{\omega_1}(p_1);\gamma_{\omega_2}(p_2);\ \cdots\ ;\gamma_{\omega_n}(p_n)\rangle=\left(\frac{g}{|\Delta|}\right)^n\langle\mathcal{Q}_{-p_1}(\Delta^{\omega_1});\mathcal{Q}_{-p_2}(\Delta^{\omega_2});\ \cdots\ ;\mathcal{Q}_{-p_n}(\Delta^{\omega_n})\rangle_0$$

where the index $\omega\in\{\mathrm{rad},\ \tan\}$ and $\Delta^{\mathrm{rad}}=\Delta,\ \Delta^{\tan}=\Delta^{\#}$. The expectation on the left hand side is integration against

$$\tfrac{1}{Z}e^{-\mathcal{V}_{\mathrm{irr}}}e^{g\int d\xi\ \bar{\mathbf{\Psi}}\gamma_s\mathbf{\Psi}}e^{-|\Delta|/g\int d\xi\ \gamma_{\mathrm{rad}}}d\mu(\gamma_s)d\mu_\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}})$$

where $\mathcal{V}_{\mathrm{irr}}$ is the irrelevant part of the interaction. When $n=2$

$$\langle\gamma_{\omega_1}(p_1);\gamma_{\omega_2}(p_2)\rangle=(2\pi)^{d+1}\delta_{\omega_1,\omega_2}\delta(p_1+p_2)+\left(\frac{g}{|\Delta|}\right)^2\langle\mathcal{Q}_{-p_1}(\Delta^{\omega_1});\mathcal{Q}_{-p_2}(\Delta^{\omega_2})\rangle_0$$

Proof: We apply the integration by parts formula

$$\int\gamma_\omega(p)A(\gamma_s)d\mu(\gamma_s)=(2\pi)^{d+1}\int\tfrac{\delta}{\delta\gamma_\omega(-p)}A(\gamma_s)d\mu(\gamma_s)$$

to obtain

$$\begin{aligned}\int\gamma_\omega(p)A(\gamma_s)&e^{-\mathcal{V}_{\mathrm{irr}}}e^{g\int d\xi\ \bar{\mathbf{\Psi}}\gamma_s\mathbf{\Psi}}e^{-|\Delta|/g\int d\xi\ \gamma_{\mathrm{rad}}}d\mu(\gamma_s)d\mu_\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}})\\
&=(2\pi)^{d+1}\int\tfrac{\delta\quad A}{\delta\gamma_\omega(-p)}(\gamma_s)e^{-\mathcal{V}_{\mathrm{irr}}}e^{g\int d\xi\ \bar{\mathbf{\Psi}}\gamma_s\mathbf{\Psi}}e^{-|\Delta|/g\int d\xi\ \gamma_{\mathrm{rad}}}d\mu(\gamma_s)d\mu_\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}})\\
&\quad+\tfrac{g}{|\Delta|}\int\mathcal{Q}_{-p}(\Delta^\omega)A(\gamma_s)e^{-\mathcal{V}_{\mathrm{irr}}}e^{g\int d\xi\ \bar{\mathbf{\Psi}}\gamma_s\mathbf{\Psi}}e^{-|\Delta|/g\int d\xi\ \gamma_{\mathrm{rad}}}d\mu(\gamma_s)d\mu_\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}})\\
&\quad-\tfrac{(2\pi)^{d+1}|\Delta|}{g}\delta(p)\int A(\gamma_s)e^{-\mathcal{V}_{\mathrm{irr}}}e^{g\int d\xi\ \bar{\mathbf{\Psi}}\gamma_s\mathbf{\Psi}}e^{-|\Delta|/g\int d\xi\ \gamma_{\mathrm{rad}}}d\mu(\gamma_s)d\mu_\Delta(\mathbf{\Psi},\bar{\mathbf{\Psi}})\end{aligned}$$

Thus

$$\langle\gamma_\omega(p)A(\gamma_s)\rangle\ =\ \left\langle(2\pi)^{d+1}\tfrac{\delta\quad A}{\delta\gamma_\omega(-p)}(\gamma_s)\right\rangle+\tfrac{g}{|\Delta|}\langle\mathcal{Q}_{-p}(\Delta^\omega)A(\gamma_s)\rangle-\tfrac{|\Delta|}{g}(2\pi)^{d+1}\delta(p)\langle A(\gamma_s)\rangle \tag{V.1}$$

Applying (V.1) with $A=1$ yields

$$\langle\gamma_\omega(p)\rangle \;=\; \tfrac{g}{|\Delta|}\left\langle \mathcal{Q}_{-p}(\Delta^\omega)\right\rangle - \tfrac{|\Delta|}{g}(2\pi)^{d+1}\delta(p) \tag{V.2}$$

Applying (V.1) with $A=\gamma_{\omega_2}(p_2)$ together with (V.2) yields

$$\begin{aligned}\langle\gamma_{\omega_1}(p_1);\gamma_{\omega_2}(p_2)\rangle &= \langle\gamma_{\omega_1}(p_1)\gamma_{\omega_2}(p_2)\rangle - \langle\gamma_{\omega_1}(p_1)\rangle\langle\gamma_{\omega_2}(p_2)\rangle \\ &= (2\pi)^{d+1}\delta_{\omega_1,\omega_2}\delta(p_1+p_2) + \tfrac{g}{|\Delta|}\Big\langle \gamma_{\omega_2}(p_2)\big(\mathcal{Q}_{-p_1}(\Delta^{\omega_1}) - \langle\mathcal{Q}_{-p_1}(\Delta^{\omega_1})\rangle\big)\Big\rangle \\ &= (2\pi)^{d+1}\delta_{\omega_1,\omega_2}\delta(p_1+p_2) + \tfrac{g^2}{|\Delta|^2}\left\langle \mathcal{Q}_{-p_1}(\Delta^{\omega_1});\mathcal{Q}_{-p_2}(\Delta^{\omega_2})\right\rangle\end{aligned} \tag{V.3}$$

upon applying (V.1) with $\mathcal{Q}_{-p_1}(\Delta^{\omega_1}) - \langle\mathcal{Q}_{-p_1}(\Delta^{\omega_1})\rangle$ in the last line.

The relationship between the ordinary and connected n point functions is given by

$$\left\langle \prod_{j=1}^n \gamma_{\omega_j}(p_j)\right\rangle = \sum_{\Pi\in\mathcal{P}_n}\prod_{\pi\in\Pi}\left\langle \prod_{j\in\pi}\big(\gamma_{\omega_j}(p_j);\big)\right\rangle$$

where $\mathcal{P}$ is the set of all partitions of $\{1,2,\cdots,n\}$. Substituting the Ansatz

$$\begin{aligned}\langle\gamma_{\omega_1}(p_1)\rangle &= \tfrac{g}{|\Delta|}\left\langle\mathcal{Q}_{-p_1}(\Delta^{\omega_1})\right\rangle - \tfrac{|\Delta|}{g}(2\pi)^{d+1}\delta(p_1) \\ \langle\gamma_{\omega_1}(p_1);\gamma_{\omega_2}(p_2)\rangle &= \tfrac{g^2}{|\Delta|^2}\left\langle\mathcal{Q}_{-p_1}(\Delta^{\omega_1});\mathcal{Q}_{-p_2}(\Delta^{\omega_2})\right\rangle + (2\pi)^{d+1}\delta_{\omega_1,\omega_2}\delta(p_1+p_2) \\ \langle\gamma_{\omega_1}(p_1);\gamma_{\omega_2}(p_2);\ \cdots\ ;\gamma_{\omega_n}(p_n)\rangle &= \tfrac{g^n}{|\Delta|^n}\left\langle\mathcal{Q}_{-p_1}(\Delta^{\omega_1});\mathcal{Q}_{-p_2}(\Delta^{\omega_2});\ \cdots\ ;\mathcal{Q}_{-p_n}(\Delta^{\omega_n})\right\rangle\end{aligned}$$

into the right hand side gives precisely the same result as repeated application of (V.1) to the left hand side. ■

We now return to power counting. By Corollary IV.5 with $r=0$ and Proposition V.1

$$n\left\langle (\gamma_{\rm rad}(0);)^{n-1}\,(\gamma_{\rm tan}(0);)^{m+1}\right\rangle = m\left\langle (\gamma_{\rm rad}(0);)^{n+1}\,(\gamma_{\rm tan}(0);)^{m-1}\right\rangle + \text{error terms}$$

When cutoffs are removed the error terms disappear. Thus, it suffices to consider generalized vertices having at least two $\gamma_{\rm rad}$ half-legs.

Once again, consider a connected vacuum graph G whose $v^{\rm th}$ generalized vertex is an n_v-loop with $n_v\ge 3$. If all the boson legs, save two per generalized vertex, are $\gamma_{\rm tan}$ legs, the power counting for G is

$$\begin{aligned}M^{(d+1)j}&\prod_v g^{n_v}|\Delta|^{2-n_v}g^{-2}\big(g^{-2}\Delta^2M^{-2j}\big)^{(n_v-2)/2}M^{(d+1)j(n_v/2-1)} \\ &= M^{(d+1)j}\prod_v M^{\frac12(d-1)jn_v-(d-1)j} \\ &= M^{(d+1)j}\prod_v M^{(d-1)j(n_v/2-1)}\end{aligned}$$

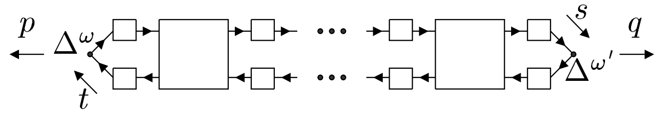
This gives positive power counting for all $n_v \geq 3,\ d \geq 2$. Consequently, the symmetry broken model is superrenormalizable and our perturbative analysis is finished.

Finally, we consider the two point expectation

$$
\begin{aligned}
\langle \gamma_\omega(p); \gamma_{\omega'}(q) \rangle &= (2\pi)^{d+1}\delta_{\omega,\omega'}\delta(p+q) + \left(\tfrac{g}{|\Delta|}\right)^2 \left\langle \mathcal{Q}_{-p}(\Delta^\omega); \mathcal{Q}_{-q}(\Delta^{\omega'}) \right\rangle_0 \\
&= (2\pi)^{d+1}\delta_{\omega,\omega'}\delta(p+q) + \left(\tfrac{g}{|\Delta|}\right)^2 \int đs\, đt \left\langle \bar{\boldsymbol{\Psi}}(t-p)\Delta^\omega \boldsymbol{\Psi}(t); \bar{\boldsymbol{\Psi}}(s-q)\Delta^{\omega'} \boldsymbol{\Psi}(s) \right\rangle_0
\end{aligned}
$$

We refer to it as the 'Goldstone Boson propagator' even though it contains both radial and tangential components of the intermediate boson field. By Theorem III.7

$$
\int đs\, đt \left\langle \bar{\boldsymbol{\Psi}}(t-p)\Delta^\omega \boldsymbol{\Psi}(t); \bar{\boldsymbol{\Psi}}(s-q)\Delta^{\omega'} \boldsymbol{\Psi}(s) \right\rangle_0 =
$$

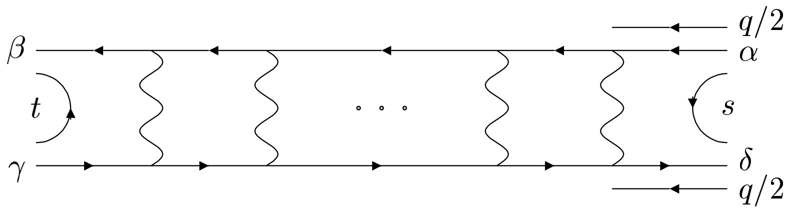


(V.4)

in which the four-legged box is the sum of all channel 2PI graphs and the two-legged box is the full two point function. Thus the ladder amputation, defined following Corollary III.6, amputates Goldstone Boson propagators.

§VI The Simple Ladder

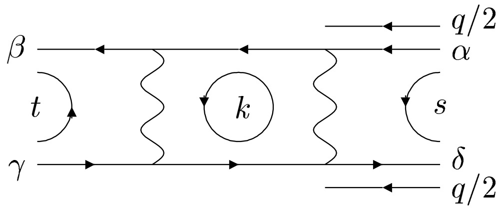
We now analyze the simple ladder



(VI.1)

Note that this Feynman diagram is drawn in terms of Nambu fields. Despite the fact that the arrows on the two sides of the ladder are pointing in opposite directions, the important components of this diagram concern the scattering, in physical fields, of Cooper pairs. If its external lines are not amputated, the ladder is a contribution to the four point function $\left\langle \boldsymbol{\Psi}_{t+\frac{q}{2},\beta} \bar{\boldsymbol{\Psi}}_{t-\frac{q}{2},\gamma} \bar{\boldsymbol{\Psi}}_{s+\frac{q}{2},\alpha} \boldsymbol{\Psi}_{s-\frac{q}{2},\delta} \right\rangle$. The $\alpha = 1$ component of the $\bar{\boldsymbol{\Psi}}_{s+\frac{q}{2},\alpha}$ combined with the $\delta = 2$ component of the $\boldsymbol{\Psi}_{s-\frac{q}{2},\delta}$ gives a Cooper pair $\bar\psi_{s+\frac{q}{2}\uparrow}\bar\psi_{-s+\frac{q}{2}\downarrow}$ of total momentum q and relative momentum s.

We wish to retain, in the evaluation of (VI.1), only the most important part of the interaction. To get some idea as to what the most important part of the interaction is, consider the second order ladder



with propagator $C(k) = \rho(\mathbf{k})[ik_0 - e(\mathbf{k})\sigma^3]^{-1}$. Here $\rho(\mathbf{k})$ just provides an ultraviolet cutoff. In this section, unlike the rest of the paper we use an ultraviolet cutoff that ignores k_0, excluding momenta k with large spatial components $\mathbf{k}$. The value of this diagram, after amputation of the external lines, is

$$
\begin{aligned}
&-\lambda^2 \int \bar{d}k \left\langle t+\tfrac{q}{2}, k-\tfrac{q}{2} \middle| V \middle| k+\tfrac{q}{2}, t-\tfrac{q}{2} \right\rangle \left\langle k+\tfrac{q}{2}, s-\tfrac{q}{2} \middle| V \middle| s+\tfrac{q}{2}, k-\tfrac{q}{2} \right\rangle (\sigma^3 C(k+\tfrac{q}{2})\sigma^3)_{\beta,\alpha} (\sigma^3 C(k-\tfrac{q}{2})\sigma^3)^t_{\gamma,\delta} \\
&= -\lambda^2 \int \bar{d}k \left\langle \tfrac{q}{2}+t, \tfrac{q}{2}-t \middle| V \middle| \tfrac{q}{2}+k, \tfrac{q}{2}-k \right\rangle \left\langle \tfrac{q}{2}+k, \tfrac{q}{2}-k \middle| V \middle| \tfrac{q}{2}+s, \tfrac{q}{2}-s \right\rangle (\sigma^3 C(k+\tfrac{q}{2})\sigma^3)_{\beta,\alpha} (\sigma^3 C(k-\tfrac{q}{2})\sigma^3)^t_{\gamma,\delta}
\end{aligned}
$$

assuming V obeys the symmetry

$$
\langle k_1, k_2 | V | k_3, k_4 \rangle = \langle k_1, -k_4 | V | k_3, -k_2 \rangle \quad .
$$

Consider the matrix element $\alpha{=}\beta{=}1, \gamma{=}\delta{=}2$. Then the propagators

$$
\begin{aligned}
(\sigma^3 C(k+q/2)\sigma^3)_{\beta,\alpha} &= \frac{\rho(\mathbf{k}+\mathbf{q}/2)}{i(k_0+q_0/2) - e(\mathbf{k}+\mathbf{q}/2)} \\
(\sigma^3 C(k-q/2)\sigma^3)^t_{\gamma,\delta} &= \frac{\rho(\mathbf{k}-\mathbf{q}/2)}{i(k_0-q_0/2) + e(\mathbf{k}-\mathbf{q}/2)}
\end{aligned}
$$

To maximize the value of the integral we need both propagators to spend as much time simultaneously near their singularities as possible. But for $k_0 + q_0/2$ and $k_0 - q_0/2$ to be simultaneously zero it is neccessary that $q_0 = 0$ and $k_0 = 0$. For $e(\mathbf{k}+\mathbf{q}/2)$ and $e(\mathbf{k}-\mathbf{q}/2)$ to be simultaneously zero $\mathbf{k}$ must be on the sphere of radius k_F centered at $-\mathbf{q}/2$ and on the sphere of radius k_F centered at $\mathbf{q}/2$. The set of such $\mathbf{k}$'s is much larger when $\mathbf{q} = 0$ than otherwise. The value of the matrix element is maximized when $q = 0$ (in fact it diverges if and only if $q = 0$) and then the dominant contributions to the integral come from k_0 near zero and $\mathbf{k}$ near the Fermi surface.

The argument of the last paragraph can be made with greater precision and generality (see [6]). The conclusion is that the most important part of the interaction is $\langle k', -k' | V | s', -s' \rangle$ with the prime signifying that

$$
k' = \left(0, \frac{\mathbf{k}}{|\mathbf{k}|} k_F\right)
$$

and s' run over the Fermi surface.

View $\langle t', -t'|V|s', -s'\rangle$ as the kernel of an integral operator on $L^2(k_F S^{d-1})$. By rotation invariance we can expand it

$$\lambda \langle t', -t'|V|s', -s'\rangle = \sum_{n\geq 0} \lambda_n \pi_n(t', s')$$

in spherical harmonics. So, $\pi_n(t', s')$ is the orthogonal projector onto the subspace of $L^2(k_F S^{d-1})$ of angular momentum n, that is, of homogeneous harmonic polynomials of degree n. We shall now restrict our attention to the case in which the coupling constant in the zero angular momentum sector, λ_0, is attractive, that is negative, and dominates the other coupling constants.

So take an interaction $\lambda \langle k_1, k_2|V|k_3, k_4\rangle = \lambda$ with $\lambda < 0$. This interaction lives purely in the angular momentum zero sector and is attractive. Then the value of a ladder (after amputation of its external lines) with n loops is

$$\Lambda_n(t, s, q) = -\lambda \Lambda(q)^n \sigma^3 \otimes \sigma^3 \tag{VI.2a}$$

where

$$\begin{aligned}\Lambda(q) &= -\lambda \int \,đk\ \left[\sigma^3 C(k+\tfrac{q}{2})\right] \otimes \left[\sigma^3 C(k-\tfrac{q}{2})^t\right] \\ &= -\lambda \int \,đk\ \left[\sigma^3 C(k+q)\right] \otimes \left[\sigma^3 C(k)^t\right]\end{aligned} \tag{VI.2b}$$

Think of q as a fixed parameter and $\Lambda(q)$ as a matrix mapping $\mathbb{C}^2 \otimes \mathbb{C}^2$ to itself. So, Λ_n is independent of s and t but has two sets of double indices $\Lambda_n(t, s, q)_{(\beta,\gamma)(\alpha,\delta)}$ with α and δ being the spinor indices of the upper and lower, respectively, external legs on the right hand side of the ladder and β and γ being the spinor indices of the external legs on the left hand side of the ladder. The n^{th} power above refers to the n^{th} power of the matrix $\Lambda(q)$.

Let us first evaluate this ladder is using the propagator

$$\begin{aligned}C(k) = C(k)^t &= \rho(\mathbf{k}) \frac{1}{ik_0 - e(\mathbf{k})\sigma^3} \\ &= -i\rho(\mathbf{k}) \begin{bmatrix} k_0 + ie(\mathbf{k}) & 0 \\ 0 & k_0 - ie(\mathbf{k}) \end{bmatrix}^{-1},\end{aligned}$$

appropriate for perturbations about the $\Delta = 0$ trivial fixed point. Since

$$(-\lambda)(-i)(-i) \int \frac{dk_0}{2\pi} \frac{1}{k_0 - ia} \frac{1}{k_0 - ib} = \lambda \frac{\operatorname{sgn}(\operatorname{Re} a)}{a-b} \begin{cases} 1 & \operatorname{Re} a \text{ and } \operatorname{Re} b \text{ of opposite sign} \\ 0 & \operatorname{Re} a \text{ and } \operatorname{Re} b \text{ of same sign} \end{cases}$$

we find that two of $\Lambda(q)$'s two eigenvectors, namely $e_1 \otimes e_2$ and $e_2 \otimes e_1$, have eigenvalue

$$-\lambda \int_{e(\mathbf{k}+\frac{\mathbf{q}}{2})e(\mathbf{k}-\frac{\mathbf{q}}{2})>0} đ\mathbf{k} \quad \frac{\rho(\mathbf{k}+\mathbf{q})\rho(\mathbf{k})}{\mp iq_0 + e(\mathbf{k}+\mathbf{q}) + e(\mathbf{k})} \operatorname{sgn} e(\mathbf{k})$$

Set $q=0$ and make the change of variables from k to $k_0, \mathbf{k}', \eta$ where

$$\begin{aligned} \mathbf{k}' &= k_F \frac{\mathbf{k}}{|\mathbf{k}|} \\ \eta &= e(\mathbf{k}) \ . \end{aligned} \tag{VI.3a}$$

Here, $\mathbf{k}'$, the projection of $\mathbf{k}$ onto the Fermi surface, runs over the sphere $k_F S^{d-1}$ and plays the rôle of all the angular variables in spherical coordinates. The rôle of the radial variable is taken by η. Note that, for $\mathbf{k}$ near the Fermi surface $\eta = \frac{1}{2m}(|\mathbf{k}|+k_F)(|\mathbf{k}|-k_F) \approx$ const $(|\mathbf{k}| - k_F)$. The measure

$$\begin{aligned} d^d\mathbf{k} &= \left(\frac{|\mathbf{k}|}{k_F}\right)^{d-1} d\mathbf{k}' d|\mathbf{k}| \\ &= \frac{m}{|\mathbf{k}|}\left(\frac{|\mathbf{k}|}{k_F}\right)^{d-1} d\mathbf{k}' d\eta \\ &= \frac{m}{k_F}\left(1+\frac{2m}{k_F^2}\eta\right)^{d/2-1} d\mathbf{k}' d\eta \end{aligned} \tag{VI.3b}$$

where the surface measure $d\mathbf{k}'$ on $k_F S^{d-1}$ is normalized so that $\int 1 \ d\mathbf{k}'$ is the surface area of $k_F S^{d-1}$.

Then, the eigenvalue is

$$\begin{aligned} -\lambda \frac{m}{(2\pi)^d k_F} & \int d\mathbf{k}' d\eta \left(1+\frac{2m}{k_F^2}\eta\right)^{d/2-1} \rho(\mathbf{k})^2 \frac{1}{2|\eta|} \\ & \geq |\lambda| \frac{m}{(2\pi)^d k_F}\left(1-\frac{2m}{k_F^2}\epsilon\right)^{d/2-1} \int_{|\eta|\leq\epsilon} d\mathbf{k}' d\eta \ \rho(\mathbf{k})^2 \frac{1}{2|\eta|} \\ & = +\infty \ . \end{aligned}$$

So $\sum_n \Lambda_n$ is a geometric series with, in this case, a ratio matrix $\Lambda(q)$ that has an eigenvalue much larger than plus one for all small q. The series diverges. The analogous calculation for the renormalization group flow shows that the Gaussian fixed point at $\Delta = 0$ is unstable.

Next consider the ladder using the propagator

$$\begin{aligned} C(k) = C(k)^t &= \rho(\mathbf{k}) \frac{1}{ik_0 - e(\mathbf{k})\sigma^3 - \Delta} \\ &= -\rho(\mathbf{k}) \frac{ik_0 + e(\mathbf{k})\sigma^3 + \Delta}{k_0^2 + E(\mathbf{k})^2}, \end{aligned}$$

appropriate for the symmetry broken model. Here $E(\mathbf{k}) = \sqrt{e(\mathbf{k})^2 + \Delta^2}$. One rung of the ladder now takes the value

$$\Lambda(q) = -\lambda \int đk \ \rho(\mathbf{k})\rho(\mathbf{k}+\mathbf{q}) \left[\sigma^3 \frac{i(k_0+q_0) + e(\mathbf{k}+\mathbf{q})\sigma^3 + \Delta}{(k_0+q_0)^2 + E(\mathbf{k}+\mathbf{q})^2}\right] \otimes \left[\sigma^3 \frac{ik_0 + e(\mathbf{k})\sigma^3 + \Delta}{k_0^2 + E(\mathbf{k})^2}\right] \tag{VI.4}$$

Define

$$\gamma = -\lambda \int d\!\!\!{}^-\mathbf{k} \ \ \frac{\rho(\mathbf{k})^2}{2E(\mathbf{k})}$$

We now prove that

Lemma VI.1 *a) The matrix norm of $\Lambda(q)$ is bounded by*

$$\|\Lambda(q)\| \le |\lambda| \int d\!\!\!{}^-\mathbf{k} \ \ \frac{\rho(\mathbf{k})^2}{|E(\mathbf{k}) + E(\mathbf{k}+\mathbf{q}) + iq_0|}$$

b) For all sufficiently small q

$$\|\Lambda(q)\| \le \gamma - c_0|\lambda|\frac{q_0^2}{\Delta^2} - \frac{2}{3}c_1|\lambda|\frac{|\mathbf{q}|^2}{\Delta^2}$$

for some nonzero constants.

c) For concreteness, take $\Delta = \Delta_1\sigma^1$. The vector $[0,1,-1,0]^{\rm t}$ is an eigenvector of $\Lambda(0)$ of eigenvalue γ. All other eigenvalues of $\Lambda(0)$ are of magnitudes strictly smaller than γ.

Remark. The vector $[0,1,-1,0]^{\rm t} = [0,1,0,0]^{\rm t} + [0,0,-1,0]^{\rm t}$. In the notation of (VI.1) the first term has $\alpha = 1,\ \delta = 2$ while the second term has $\alpha = 2,\ \delta = 1$. Thus the dominant eigenvector corresponds to a Cooper pair of momentum zero.

Proof of a): Because we have chosen an ultraviolet cutoff that doesn't involve k_0, the k_0 integral can be done explicitly by contour integration. There are two poles in the upper half plane, with $k_0 = -q_0 + iE(\mathbf{k}+\mathbf{q})$ and $k_0 = +iE(\mathbf{k})$ respectively, so that

$$\begin{aligned}
&\int \frac{dk_0}{2\pi} \left[\sigma^3 \frac{i(k_0+q_0) + e(\mathbf{k}+\mathbf{q})\sigma^3 + \Delta}{(k_0+q_0)^2 + E(\mathbf{k}+\mathbf{q})^2}\right] \otimes \left[\sigma^3 \frac{ik_0 + e(\mathbf{k})\sigma^3 + \Delta}{k_0^2 + E(\mathbf{k})^2}\right] \\
&= \left[\sigma^3 \frac{-E(\mathbf{k}+\mathbf{q}) + e(\mathbf{k}+\mathbf{q})\sigma^3 + \Delta}{2E(\mathbf{k}+\mathbf{q})}\right] \otimes \left[\sigma^3 \frac{-iq_0 - E(\mathbf{k}+\mathbf{q}) + e(\mathbf{k})\sigma^3 + \Delta}{(-q_0 + iE(\mathbf{k}+\mathbf{q}))^2 + E(\mathbf{k})^2}\right] \\
&\qquad\qquad + \left[\sigma^3 \frac{iq_0 - E(\mathbf{k}) + e(\mathbf{k}+\mathbf{q})\sigma^3 + \Delta}{(q_0 + iE(\mathbf{k}))^2 + E(\mathbf{k}+\mathbf{q})^2}\right] \otimes \left[\sigma^3 \frac{-E(\mathbf{k}) + e(\mathbf{k})\sigma^3 + \Delta}{2E(\mathbf{k})}\right] \\
&= \frac{1}{(E+E_{\mathbf{q}})^2 + q_0^2}\sigma^3 \otimes \sigma^3 \left\{ \frac{(E+E_{\mathbf{q}})}{2} \left[-\mathbb{1}\otimes\mathbb{1} + \left(\frac{e_q}{E_q}\sigma^3 + \frac{\Delta}{E_q}\right) \otimes \left(\frac{e}{E}\sigma^3 + \frac{\Delta}{E}\right)\right]\right. \\
&\qquad\qquad\qquad \left. + i\frac{q_0}{2}\left[\mathbb{1}\otimes\left(\frac{e}{E}\sigma^3 + \frac{\Delta}{E}\right) - \left(\frac{e_q}{E_q}\sigma^3 + \frac{\Delta}{E_q}\right)\otimes\mathbb{1}\right]\right\} \\
&= \frac{1}{(E+E_{\mathbf{q}})^2 + q_0^2} \left\{ \frac{(E+E_{\mathbf{q}})}{2} \left[-\sigma^3\otimes\sigma^3 + \left(\frac{e_q}{E_q}\mathbb{1} + i\frac{\Delta^\sharp}{E_q}\right) \otimes \left(\frac{e}{E}\mathbb{1} + i\frac{\Delta^\sharp}{E}\right)\right]\right. \\
&\qquad\qquad\qquad \left. + i\frac{q_0}{2}\left[\sigma^3\otimes\left(\frac{e}{E}\mathbb{1} + i\frac{\Delta^\sharp}{E}\right) - \left(\frac{e_q}{E_q}\mathbb{1} + i\frac{\Delta^\sharp}{E_q}\right)\otimes\sigma^3\right]\right\}
\end{aligned} \tag{VI.5}$$

where we use E, e, E_q and $e_{\mathbf{q}}$ as short forms for $E(\mathbf{k}), e(\mathbf{k}), E(\mathbf{k}+\mathbf{q})$ and $e(\mathbf{k}+\mathbf{q})$ respectively.

The four 4×4 matrices

$$\mathbb{1}\otimes\mathbb{1} \qquad \left(\frac{e_q}{E_q}\sigma^3+\frac{\Delta}{E_q}\right)\otimes\left(\frac{e}{E}\sigma^3+\frac{\Delta}{E}\right) \qquad \mathbb{1}\otimes\left(\frac{e}{E}\sigma^3+\frac{\Delta}{E}\right) \qquad \left(\frac{e_q}{E_q}\sigma^3+\frac{\Delta}{E_q}\right)\otimes\mathbb{1}$$

are mutually commuting and self-adjoint. Furthermore, since $\frac{e}{E}\sigma^3+\frac{\Delta}{E}$ is traceless with determinant $-\frac{e^2+\Delta^2}{E^2}=-1$, the eigenvalues of the matrix inside the brackets $\{\quad\}$ of the second last line of (VI.5) are, for each $\mathbf{k},\mathbf{q}$,

$$0 \qquad 0 \qquad -(E+E_q)-iq_0 \qquad -(E+E_q)+iq_0$$

Consequently, the norm of the matrix inside the brackets $\{\quad\}$ is at most $|E+E_q+iq_0|$ and

$$\begin{aligned}
\|\Lambda(q)\| &\le |\lambda|\int d\!\!\!{}^-\mathbf{k}\ \frac{\rho(\mathbf{k})\rho(\mathbf{k}+\mathbf{q})}{|E(\mathbf{k})+E(\mathbf{k}+\mathbf{q})+iq_0|}\\
&\le |\lambda|\int d\!\!\!{}^-\mathbf{k}\ \frac{\frac{1}{2}\rho(\mathbf{k})^2+\frac{1}{2}\rho(\mathbf{k}+\mathbf{q})^2}{|E(\mathbf{k})+E(\mathbf{k}+\mathbf{q})+iq_0|}\\
&= |\lambda|\int d\!\!\!{}^-\mathbf{k}\ \frac{\rho(\mathbf{k})^2}{|E(\mathbf{k})+E(\mathbf{k}+\mathbf{q})+iq_0|}
\end{aligned}$$

where we have used the change of variables $\mathbf{k}\to-\mathbf{k}-\mathbf{q}$ in the second half of the integral.

Proof of b): We Taylor expand

$$L(q)=|\lambda|\int d\!\!\!{}^-\mathbf{k}\ \frac{\rho(\mathbf{k})^2}{|E(\mathbf{k})+E(\mathbf{k}+\mathbf{q})+iq_0|}$$

to second order in q. Since

$$\frac{\partial}{\partial q_0}\left[\big(E(\mathbf{k})+E(\mathbf{k}+\mathbf{q})\big)^2+q_0^2\right]^{-1/2}=-\frac{q_0}{\left[\big(E(\mathbf{k})+E(\mathbf{k}+\mathbf{q})\big)^2+q_0^2\right]^{3/2}}$$

we have that

$$\frac{\partial L}{\partial q_0}(0)=0$$
$$\frac{\partial^2\ L}{\partial q_0\partial q_i}(0)=0 \qquad i\neq 0$$

and

$$\frac{\partial^2 L}{\partial q_0^2}(0)=-|\lambda|\int d\!\!\!{}^-\mathbf{k}\ \frac{\rho(\mathbf{k})^2}{8E(\mathbf{k})^3}<0\ .$$

Before moving on to the spatial directions we show that $\frac{\partial^2 L}{\partial q_0^2}(0)$ is $O(\frac{\lambda}{\Delta^2})$. Suppose that the ultraviolet cutoff restricts $\mathbf{k}$ to $\{\mathbf{k} \mid |e(\mathbf{k})| \le 1\}$ and make the change of variables (VI.3), followed by $\eta = |\Delta|\zeta$ where $|\Delta| = \sqrt{\Delta_1^2 + \Delta_2^2}$:

$$\begin{aligned}
|\lambda| \int d\!\!{}^-\mathbf{k}\ \frac{\rho(\mathbf{k})^2}{8E(\mathbf{k})^3} &= |\lambda| \frac{m}{(2\pi)^d k_F} \int_{|\eta|\le 1} d\mathbf{k}' d\eta \left(1 + \frac{2m}{k_F^2}\eta\right)^{d/2-1} \frac{1}{8(\eta^2+\Delta^2)^{3/2}} \\
&= \frac{|\lambda|}{\Delta^2} \frac{m}{(2\pi)^d k_F} \int_{|\zeta|\le 1/|\Delta|} d\mathbf{k}' d\zeta \left(1 + \frac{2m|\Delta|}{k_F^2}\zeta\right)^{d/2-1} \frac{1}{8(\zeta^2+1)^{3/2}} \\
&= \frac{|\lambda|}{\Delta^2} \frac{m}{(2\pi)^d k_F} \mathrm{Vol}(k_F S^{d-1}) \int_{|\zeta|\le 1/|\Delta|} d\zeta \left(1 + \frac{2m|\Delta|}{k_F^2}\zeta\right)^{d/2-1} \frac{1}{8(\zeta^2+1)^{3/2}}
\end{aligned}$$

As $|\Delta| \to 0$, the main contribution to the integral is

$$\begin{aligned}
\lim_{|\Delta|\to 0} \int_{|\zeta|\le 1/|\Delta|} d\zeta \frac{1}{8(\zeta^2+1)^{3/2}} &= \int_{-\infty}^{\infty} d\zeta \frac{1}{8(\zeta^2+1)^{3/2}} \\
&= \frac{1}{4}
\end{aligned}$$

The rest of the integral is bounded by

$$\int_{|\zeta|\le 1/|\Delta|} d\zeta \left| \left(1 + \frac{2m|\Delta|}{k_F^2}\zeta\right)^{d/2-1} - 1 \right| \frac{1}{8(\zeta^2+1)^{3/2}} \le \text{const} \int_0^\infty d\zeta \frac{2m|\Delta|}{k_F^2} \frac{\zeta}{8(\zeta^2+1)^{3/2}}$$

by the mean value theorem. Hence it is down by a factor of $O(|\Delta|)$.

For spatial directions $i = 1, 2, 3$ the first derivative, with $q_0 = 0$,

$$\frac{\partial}{\partial q_i} \frac{1}{E(\mathbf{k}) + E(\mathbf{k}+\mathbf{q})} = -\frac{1}{\big(E(\mathbf{k}) + E(\mathbf{k}+\mathbf{q})\big)^2} \frac{e(\mathbf{k}+\mathbf{q})}{E(\mathbf{k}+\mathbf{q})} \frac{\mathbf{k}_i + \mathbf{q}_i}{m}$$

leads, to the second derivative

$$\frac{\partial^2}{\partial q_i \partial q_j} \frac{1}{E(\mathbf{k}) + E(\mathbf{k}+\mathbf{q})} \bigg|_{q\,=\,0} = -\delta_{i,j} \frac{e(\mathbf{k})}{4E(\mathbf{k})^3} \frac{1}{m} + \frac{e(\mathbf{k})^2 - \Delta^2}{4E(\mathbf{k})^5} \frac{\mathbf{k}_i}{m} \frac{\mathbf{k}_j}{m}$$

so that

$$\frac{\partial^2\ L}{\partial q_i \partial q_j}(0) = |\lambda| \delta_{i,j} \int d\!\!{}^-\mathbf{k}\ \rho(\mathbf{k})^2 \left\{ -\frac{e(\mathbf{k})}{4mE(\mathbf{k})^3} + \frac{e(\mathbf{k})^2 - \Delta^2}{4E(\mathbf{k})^5} \frac{\mathbf{k}_i^2}{m^2} \right\}$$

Again make the change of variables (VI.3) followed by the scaling $\eta = |\Delta|\zeta$. For $|\Delta|$ small, and it is very small indeed, the dominant contribution to the integral comes from $e(\mathbf{k}) \approx 0$. As in the calculation above, we can replace the Jacobean from the change of variables and k_i^2 by positive constants without affecting the leading contribution. Then the integral becomes

$$\begin{aligned}
&|\lambda|\delta_{i,j}\int_{|\eta|\le 1} d\eta\ \left\{-c_2\frac{\eta}{(\eta^2+\Delta^2)^{3/2}}+c_1\frac{\eta^2-\Delta^2}{(\eta^2+\Delta^2)^{5/2}}\right\}\\
&=|\lambda|\delta_{i,j}\int_{|\zeta|\le 1/|\Delta|} d\zeta\ \left\{-\frac{c_2}{|\Delta|}\frac{\zeta}{(\zeta^2+1)^{3/2}}+\frac{c_1}{\Delta^2}\frac{\zeta^2-1}{(\zeta^2+1)^{5/2}}\right\}\\
&\approx|\lambda|\delta_{i,j}\int_{-\infty}^{\infty} d\zeta\ \left\{-\frac{c_2}{|\Delta|}\frac{\zeta}{(\zeta^2+1)^{3/2}}+\frac{c_1}{\Delta^2}\frac{\zeta^2-1}{(\zeta^2+1)^{5/2}}\right\}\\
&=-\frac{2}{3}|\lambda|\delta_{i,j}\frac{c_1}{\Delta^2}
\end{aligned}$$

for $|\Delta|$ sufficiently small.

Proof of c): Using the notation

$$A\otimes B=\begin{bmatrix}a_{1,1}B & a_{1,2}B\\ a_{2,1}B & a_{2,2}B\end{bmatrix}$$

we have

$$-\sigma^3\otimes\sigma^3=\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}$$

$$\left(\frac{e_q}{E_q}\mathbb{1}+i\frac{\Delta^\sharp}{E_q}\right)\otimes_s\left(\frac{e}{E}\mathbb{1}+i\frac{\Delta^\sharp}{E}\right)=\frac{1}{EE_q}\begin{bmatrix}ee_q & \frac{e+e_q}{2}\Delta_1 & \frac{e+e_q}{2}\Delta_1 & \Delta_1^2\\ -\frac{e+e_q}{2}\Delta_1 & ee_q & -\Delta_1^2 & \frac{e+e_q}{2}\Delta_1\\ -\frac{e+e_q}{2}\Delta_1 & -\Delta_1^2 & ee_q & \frac{e+e_q}{2}\Delta_1\\ \Delta_1^2 & -\frac{e+e_q}{2}\Delta_1 & -\frac{e+e_q}{2}\Delta_1 & ee_q\end{bmatrix}$$

$$\sigma^3\otimes\left(\frac{e}{E}\mathbb{1}+i\frac{\Delta^\sharp}{E}\right)-\left(\frac{e}{E}\mathbb{1}+i\frac{\Delta^\sharp}{E}\right)\otimes\sigma^3=\frac{1}{E}\begin{bmatrix}0&\Delta_1&-\Delta_1&0\\-\Delta_1&2e&0&\Delta_1\\\Delta_1&0&-2e&-\Delta_1\\0&-\Delta_1&\Delta_1&0\end{bmatrix}$$

The expressions $e_q, E-q, e$ and E appear in different places here than in (VI.5). That's permissable because we can make judicious use of the change of variables $\mathbf{k}\to-\mathbf{k}-\mathbf{q}$ to symmetrize the integrand. In the basis

$$\mathbf{e}_1=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\qquad\mathbf{e}_2=\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\qquad\mathbf{e}_3=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\qquad\mathbf{e}_4=\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

the matrices become

$$-\sigma^3\otimes\sigma^3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left(\frac{e_q}{E_q}1\!\!1 + i\frac{\Delta^\sharp}{E_q}\right)\otimes_s\left(\frac{e}{E}1\!\!1 + i\frac{\Delta^\sharp}{E}\right) = \frac{1}{EE_q}\begin{bmatrix} ee_q+\Delta_1^2 & 0 & 0 & 0 \\ 0 & ee_q-\Delta_1^2 & (e+e_q)\Delta_1 & 0 \\ 0 & -(e+e_q)\Delta_1 & ee_q-\Delta^2 & 0 \\ 0 & 0 & 0 & ee_q+\Delta_1^2 \end{bmatrix}$$

$$\sigma^3\otimes\left(\frac{e}{E}1\!\!1+i\frac{\Delta^\sharp}{E}\right)-\left(\frac{e}{E}1\!\!1+i\frac{\Delta^\sharp}{E}\right)\otimes\sigma^3=\frac{2}{E}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_1 \\ 0 & 0 & 0 & e \\ 0 & -\Delta_1 & e & 0 \end{bmatrix}$$

Still in basis $\mathbf{e}_i,\ 1\le i\le 4$, but specializing to $q=0$ we have

$$\Lambda(0) = -\lambda\int d\!\!{}^-\mathbf{k}\ \rho(\mathbf{k})^2\frac{1}{2E(\mathbf{k})}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\Delta_1^2/E^2 & e\Delta_1/E^2 & 0 \\ 0 & -e\Delta_1/E^2 & e^2/E^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so that $\mathbf{e}_1$ is an eigenvector of eigenvalue 0 and $\mathbf{e}_4$ is an eigenvector of eigenvalue γ.

The matrix

$$\begin{aligned} A(\mathbf{p}) &= \begin{bmatrix} -\Delta_1^2/E(\mathbf{p})^2 & e(\mathbf{p})\Delta_1/E(\mathbf{p})^2 \\ -e(\mathbf{p})\Delta_1/E(\mathbf{p})^2 & e(\mathbf{p})^2/E(\mathbf{p})^2 \end{bmatrix} \\ &= \frac{1}{E(\mathbf{p})^2}\begin{bmatrix} \Delta_1 \\ e(\mathbf{p}) \end{bmatrix}\begin{bmatrix} -\Delta_1 & e(\mathbf{p}) \end{bmatrix} \end{aligned}$$

has eigenvalues 0 and $(e^2-\Delta^2)/E^2$ with corresponding eigenvectors $[e,\Delta_1]^{\mathrm{t}}$ and $[\Delta_1,e]^{\mathrm{t}}$. Because

$$A^\dagger(\mathbf{p})A(\mathbf{k}) = \frac{e(\mathbf{p})e(\mathbf{k})+\Delta_1^2}{E(\mathbf{p})^2E(\mathbf{k})^2}\begin{bmatrix} -\Delta_1 \\ e(\mathbf{p}) \end{bmatrix}\begin{bmatrix} -\Delta_1 & e(\mathbf{k}) \end{bmatrix}$$

is of norm $\frac{e(\mathbf{p})e(\mathbf{k})+\Delta_1^2}{E(\mathbf{p})E(\mathbf{k})}$, which is strictly less than 1 except on the set of measure zero $\{\ (\mathbf{p},\mathbf{k})\mid e(\mathbf{p})=e(\mathbf{k})\ \}$, the norm

$$\begin{aligned} \left\|\lambda\int d\!\!{}^-\mathbf{k}\ \rho(\mathbf{k})^2\frac{1}{2E(\mathbf{k})}A(k)\right\|^2 &= \lambda^2\left\|\int d\!\!{}^-\mathbf{k}\,d\!\!{}^-\mathbf{p}\ \rho(\mathbf{k})^2\rho(\mathbf{p})^2\frac{1}{4E(\mathbf{k})E(\mathbf{p})}A^\dagger(\mathbf{k})A(\mathbf{p})\right\| \\ &< \lambda^2\int d\!\!{}^-\mathbf{k}\,d\!\!{}^-\mathbf{p}\ \rho(\mathbf{k})^2\rho(\mathbf{p})^2\frac{1}{4E(\mathbf{k})E(\mathbf{p})} = \gamma^2 \end{aligned}$$

■

We just showed in Lemma VI.3 that show, for q small,

$$\|\Lambda(q)\| \leq \gamma - c_0|\lambda|\frac{q_0^2}{\Delta^2} - \frac{2}{3}c_1|\lambda|\frac{|\mathbf{q}|^2}{\Delta^2} \tag{VI.6}$$

for some nonzero constants. This then implies that the full ladder obeys

$$\left|\sum_{n=1}^{\infty}\Lambda_n(t,s,q)\right| \leq \frac{|\lambda|}{1-\gamma+|\lambda|(c_0q_0^2+\frac{2}{3}c_1|\mathbf{q}|^2)/\Delta^2}$$

The BCS equation (II.6) tells us that, to first order in λ, $\gamma = 1$. But, as was shown in §III, the (amputated) four point function as the sum of generalized ladders whose "rungs" consist of all channel two particle irreducible four point functions. The "rung" $\Lambda(q)$ is just the first order contribution to the generalized rung. The BCS equation should be interpreted as putting the above bound exactly on the radius of convergence of the geometric series when $q = 0$ so that

$$\left|\sum_{n=1}^{\infty}\Lambda_n(t,s,q)\right| \leq \frac{1}{(c_0q_0^2+\frac{2}{3}c_1|\mathbf{q}|^2)/\Delta^2} \tag{VI.7}$$

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