Two Dimensional Many Fermion Systems as Vector Models

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Abstract We show that, in two space dimensions, the full many Fermion interaction has the structure of a vector model at every energy scale. The "colors" arise from a decomposition of the Fermi circle into sectors of length set by the energy scale. These sectors play an essential role in summing up the complete perturbation series. The three dimensional vertex is intermediate between vector and matrix models. In any number of dimensions the reduced interaction is a vector model.

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§I Introduction

In [1a] we considered a $d \geq 2$ dimensional many Fermion interaction whose $\ell = 0$ angular momentum sector is attractive and dominant. We showed that the effective vertex in the symmetry breaking energy regime $\left(k_0^2 + e(\mathbf{k})^2\right)^{1/2} \approx \Delta$, where Δ is the BCS gap and $e(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - \mu$, has the structure of an N component vector model with $N \approx \Delta^{-(d-1)}$. We also verified that bubbles are neutral in powers of $1/N$ and that all other Fermion loops are proportional to strictly positive powers of $1/N$. This observation justifies the usual one loop BCS gap equation.

Here we show, using the results of [1b], that in two space dimensions the full interaction has the structure of a vector model at *all* energy scales. This structure was used in [1b] to show that the sum all of perturbation theory down to energy scale $\approx \Delta$ converges for all bare coupling constants $|\lambda| \leq \text{const}$ where const is a strictly positive constant *independent* of Δ. We will also draw attention to the difference between two and three dimensions.

We consider the formal action $\mathcal{A}(\psi, \bar\psi) = -\int đk\, \left(ik_0 e(\mathbf{k})\right)\bar\psi(k)\psi(k) - \mathcal{V}(\psi, \bar\psi)$ in which $k = (k_0, \mathbf{k}) \in \mathbb{R}^{d+1}$ and $đk = \frac{d^{d+1}k}{(2\pi)^{d+1}}$. The interaction is

$$\mathcal{V}(\psi, \bar\psi) = \tfrac{\lambda}{2} \int \prod_{i=1}^{4} đk_i\ (2\pi)^{d+1}\delta(k_1+k_2-k_3-k_4)\ \bar\psi(k_1)\psi(k_3)\,\langle k_1, k_2|\mathrm{V}|k_3, k_4\rangle\,\bar\psi(k_2)\psi(k_4) \tag{I.1}$$

Here, $k' = (0, \frac{\mathbf{k}}{|\mathbf{k}|}\sqrt{2\mathbf{m}\mu})$ is the projection of k onto the Fermi surface. In the above expressions, the electron fields are vectors $\psi(k) = \begin{pmatrix}\psi_\uparrow(k)\\ \psi_\downarrow(k)\end{pmatrix}$ and $\bar\psi(k) = \left(\bar\psi_\uparrow(k), \bar\psi_\downarrow(k)\right)$ whose components $\psi_\sigma(k)$, $\bar\psi_\sigma(k)$ generate an infinite dimensional Grassmann algebra over $\mathbb{C}$. Their covariance is

$$C(\xi_1, \xi_2) = \delta_{\sigma_1,\sigma_2} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{e^{i\langle p\,,\,\xi_1-\xi_2\rangle_-}}{ip_0 - e(\mathbf{p})}, \text{ where } \langle p, \xi\rangle_- \;=\; -p_0 t + \mathbf{p}\cdot\mathbf{x} \tag{I.2}$$

§II Slices and Sectors

To investigate the long range behavior of correlation functions at low temperature, it is natural to use a renormalization group analysis ([2-3]) near the Fermi surface. To do this, we slice the covariance (free propagator) around its singularity on the Fermi sphere.

To analyze a Euclidean field theory, one defines a momentum k to be of scale j if $|k| \approx 2^j$. As $j \to \infty$ (respectively $j \to -\infty$) the momentum k approachs the ultraviolet (respectively infrared) end of the model.

In non-relativistic solid state physics the natural scales consist of finer and finer shells around the Fermi surface. For each negative integer $j = 0, -1, -2, \ldots$ the j-th slice contains all momenta in a shell of thickness 2^j a distance 2^j from the singular locus $\{\ k \in \mathbb{R}^{d+1}\ \big|\ k_0 = 0,\ |\mathbf{k}| = \sqrt{2\mathbf{m}\mu}\ \}$. The propagator for the j-th slice is

$$C^j(\xi_1, \xi_2) = \delta_{\sigma_1,\sigma_2} \int đk \frac{e^{i\langle k, \xi_1 - \xi_2\rangle_-}}{ik_0 - e(\mathbf{k})} 1_j\left(k_0^2 + e(\mathbf{k})^2\right) \tag{II.1}$$

where $1_j\big(k_0^2 + e(\mathbf{k})^2\big)$ is the characteristic function for the set $2^j \le |ik_0 - e(\mathbf{k})| < 2^{j+1}$. (A smooth partition of unity is required for a technically correct analysis). The infrared propagator is $C(\xi_1, \xi_2) = \sum\limits_{j \le 0} C_j(\xi_1, \xi_2)$. The Schwinger functions are obtained by assigning each line of each Feynman diagram a scale j, then summing over all such assignments.

Each single scale propagator (II.1) is supported in momentum space on a $d+1$ dimensional manifold with boundary. The natural coordinates for this manifold are k_0, $\eta = e(\mathbf{k})$ and $\mathbf{k}' = \sqrt{2\mathbf{m}\mu}\frac{\mathbf{k}}{|\mathbf{k}|}$. In these coordinates the shell is $\{\ k\ \big|\ 2^j \le \sqrt{k_0^2 + \eta^2} \le \text{const}\, 2^j\ \}$ and is topologically $S^{d-1} \times S^1 \times [0,1]$. The first factor, the Fermi sphere S^{d-1}, has a macroscopic diameter of order 1 while S^1 and $[0,1]$ have small diameters of order 2^j.

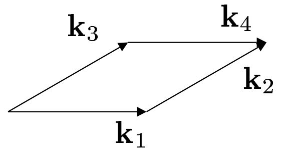
The fact that this manifold has two length scales, 1 and 2^j, of radically different size reflects the basic anisotropy between frequency k_0 and momentum $\mathbf{k}$. Consequently the behavior of $C^j(0,\xi)$ at large $\xi = (\mathbf{x}, t)$ cannot be simply characterized as 'decay at length scale 2^{-j}'. Rather, smoothly cutoff, C^j obeys $\left|C^j(0,\xi)\right| \le \text{const}\, 2^j \left[1 + |\mathbf{x}|\right]^{(1-d)/2} \left[1 + 2^j|\xi|\right]^{-n}$.

To obtain regions in momentum space all of whose dimensions are of order 2^j, it is natural to further divide the j^{th} shell, through a partition of unity, into $N_j = 2^{-(d-1)j}$ pieces, each having longest and shortest diameters of order 2^j . We call each piece an isotropic sector. These sectors function as colors in a many component model. We found that, while they are unnecessary for bounding a single graph, they are essential for summing up perturbation theory. They are related to the patches in Haldane's innovative, phenomenological theory of the Fermi surface.

We now discuss a basic question. Every interaction vertex couples four fields, each of which has a sector index. How many 4-tuples of sector indices are coupled?

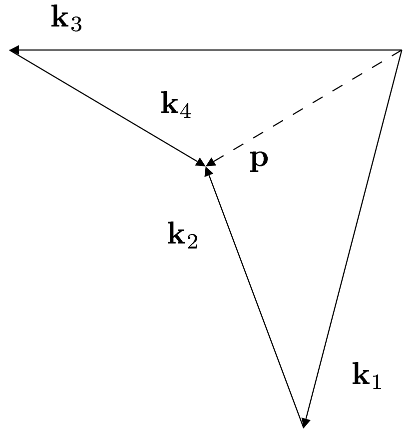
The momenta $k_1,\ k_2,\ k_3$ and k_4 of the four fields coupled by the vertex (I.1) are constrained by conservation of momentum. If all of these momenta are of scale $j \ll 0$, then all of their 0^{th} components are very close to zero and all of their lengths are very close to $\sqrt{2\mathbf{m}\mu}$. Generically, these are the only constraints relating $k_1,\ k_2,\ k_3$ and k_4.

In two space dimensions four momenta $\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k}_4$ of very nearly the same length which obey $\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 = 0$ form an approximate parallelogram.



The directions of two sides of a parallelogram may be chosen arbitrarily. Once they have been chosen the directions of the remaining two sides are completely determined. Thus roughly speaking, in two space dimensions, the sector indices of two momenta may be chosen arbitrarily and once they have been chosen the sector indices of the remaining two momenta are fixed. As there are 2^j different possible values of the sector index in two space dimensions, the interaction (I.1) couples roughly 2^{2j} four-tuples of sector indices at scale j in two space dimensions.

In $d > 2$ space dimensions, the sectors of $\mathbf{k}_1$ and $\mathbf{k}_2$ can again be chosen arbitrarily. There are $2^{-2(d-1)j}$ possibilities. Once $\mathbf{k}_1$ and $\mathbf{k}_2$ are chosen, the sum $\mathbf{p} = \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$ is fixed. But the triangle with sides $\mathbf{k}_3,\ \mathbf{k}_4$ and $\mathbf{p}$ is still free to rotate about the fixed $\mathbf{p}$.



The set of possible orientations of this triangle is S^{d-2}. Crudely speaking this accounts for another $2^{-(d-2)j}$ possible sector choices. This calculation suggests that (I.1) couples on the order of $2^{-(3d-4)j}$ four-tuples of sectors.

The following lemma, proven in [1b], gives the precise result that is motivated by the argument above. For a rigorous construction it is necessary to work in finite volume. Then, momentum is not exactly conserved. That is, $|\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4| \le m2^j$ for some $m>0$. The lemma also takes this into account.

Lemma. *Let $m \ge 1$. The number of four-tuples $\{S_1,S_2,S_3,S_4\}$ of sectors of scale j for which there exist $\mathbf{k}_i \in \mathrm{I\!R}^d$, $i=1,2,3,4$ satisfying $\mathbf{k}'_i \in S_i$, $|\mathbf{k}_i-\mathbf{k}'_i| \le \mathrm{const}\, 2^j$ and $|\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4| \le m2^j$ is bounded by $\mathrm{const}\, m^d 2^{(-3d+4)j}(1+|j|\delta_{d,2})$.*

Thus, in two dimensions, the number of four-tuples of sectors coupled at scale j, with approximate conservation of momentum, is about N_j^2 where $N_j = 2^{-j}$ is the number of sectors. There is a logarithmic correction, $|j|$, that is generated when the "parallelogram" almost collapses to a line. This logarithm can be accommodated constructively (see §III).

Recall that a vector model $(\vec{\phi}\cdot\vec{\phi})^2$ with N colours couples N^2 four-tuples of colours. The color index of one ϕ in a dot product determines the index of the other ϕ. The Lemma shows that the two dimensional vertex at scale j conserves sector indices (up to corrections lower order in N_j) just as an N_j-vector model conserves color. Thus, the two dimensional vertex has the structure of a vector model. We expect that this remark applies in other two dimensional solid state physics problems, for example, localization of a single two dimensional electron in a weak random potential.

In three dimensions, the full vertex does not conserve sector index. Now, in addition to the usual sector indices of $\mathbf{k}_1$ and $\mathbf{k}_2$, the angle between the triangles $\mathbf{k}_1\mathbf{k}_2\mathbf{p}$ and $\mathbf{k}_3\mathbf{k}_4\mathbf{p}$ may also be chosen freely. The vertex is "twisted". By the Lemma, the number of coupled sector four-tuples is 2^{-5j}. Since, the total number of sectors in three dimensions is $N_j = 2^{-2j}$, we have $2^{-5j} = N_j^{5/2}$. This count is intermediate between that for an N vector model $(\vec{\phi}\cdot\vec{\phi})^2$, which couples N^2 four-tuples, and that for a matrix model such as $Tr\Phi^4$ (where Φ is an $N\times N$ matrix), which couples N^4 four-tuples.

This discussion applies to generic vertices, but in any dimension there are special interactions with vector type sector conservation laws, for example the reduced BCS interaction for Cooper pairs

$$\lambda \int_{|q|<\mathrm{const}\,2^j} đq\, đt\, đs\ \ \langle \mathbf{t}, -\mathbf{t}|V|\mathbf{s}, -\mathbf{s}\rangle\, \bar\psi_\alpha(t+\tfrac{q}{2})\bar\psi_\beta(-t+\tfrac{q}{2})\psi_\beta(-s+\tfrac{q}{2})\psi_\alpha(s+\tfrac{q}{2})\ .$$

This vector structure of the s-wave vertex is discussed in [1a]. More generally any vertex that forces the four momenta to lie in a (possibly vertex dependent) plane has the structure of a vector model.

§III Sectors and the Convergence of Many Fermion Perturbation Series

In this section we compare the size of a single graph of order n with the size of the sum of all graphs of order n. A complete discussion requires a full renormalization group analysis at all scales. Here, to concentrate on the large orders of perturbation theory, rather than on the flow arising from low orders of perturbation theory, we fix a scale j.

Crude determinant bounds imply that the Schwinger functions for a theory at scale j with propagator C^j given by (II.1) and interaction $\lambda\mathbf{V} = \mathcal{V}$ are analytic in λ around $\lambda = 0$. Does the radius of analyticity, r_j, remain bounded away from zero as $j \to -\infty$? Control of r_j is essential in any nonperturbative program to analyze the renormalization group flow. We have

Theorem. [1b] *In two space dimensions* $r_j \geq \mathrm{const} > 0$

So, for a weakly coupled two dimensional model, with a circular Fermi surface, there are no exotic effects due to high orders of perturbation theory. Presently, there is no analogue of this theorem in three dimensions. Our best estimate, to date, is $r_j \geq \mathrm{const}\, 2^{j/2}$.

Sectors play an essential rôle in proving this Theorem. Indeed power counting estimates for a single graph of given order (perturbative power counting) differ from power counting estimates for the *sum* of all graphs of that order (constructive power counting). Let $G_{n,p}$ be a connected graph of scale j and order n with $2p$ external lines. The number l of lines and L of independent loops in this graph are $l = 2n - p$ and $L = l - n + 1$. Write C_j

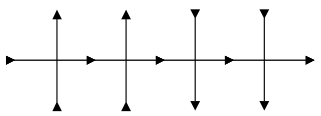
as the sum of $2^{-(d-1)j}$ single sector propagators and smooth off the characteristic function 1_j. Each single sector propagator obeys the position space bound

$$
\begin{aligned}
|C^{j,s}(\xi_1,\xi_2)| &= \delta_{\sigma_1,\sigma_2}\left|\int đp\frac{e^{i\langle p,\xi_1-\xi_2\rangle_-}}{ip_0-e(\mathbf{p})}1_j(p_0^2+e(\mathbf{p})^2)g_s(\mathbf{p})\right| \\
&\le \mathrm{const}\, 2^{(d+1)j}2^{-j}[1+2^j|\xi_1-\xi_2|]^{-100} \qquad \text{(III.1)}
\end{aligned}
$$

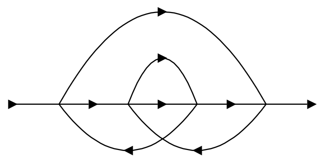
The first factor is the volume of integration. The second is the supremum of the integrand. The third is derived through multiple integration by parts. Thus, when $G_{n,p}$ is evaluated in position space, there is one factor of 2^{dj} per line. Furthermore, the position of one vertex is held fixed and the positions of the remaining $n-1$ vertices are integrated over $\mathbb{R}^{d+1}$. Since each such integral is controlled using a function that decays at rate 2^j each such integral yields $2^{-(d+1)j}$. Finally the sector index of each propagator must be summed over. But, by conservation of momentum, there really are only L independent sector index sums. So the bound is $2^{ldj}2^{-(n-1)(d+1)j}2^{-L(d-1)j} = 2^{j(2-p)}$. This perturbative power counting is independent of dimension and typical of strictly renormalizable theories like ϕ_4^4.

Ideally, the sum of all n^{th} order graphs would also be bounded by $\mathrm{const}^{\,n}2^{j(2-p)}$. However, the argument given above does not apply to the sum of all graphs because they do not share a fixed loop structure.

Constructive methods for summing up graphs performs, roughly speaking, the sum over connected graphs in two steps [2a]. First one sums over all trees. Then one sums over graphs having this tree as a spanning tree. This second step consists of choosing "Wick contractions" to add loops to the spanning tree.



a linear spanning tree



a Wick contraction scheme

The choice of tree, by Cayley's theorem, is bounded by n^{n-2} and is paid for (up to e^n) by the symmetry factor $1/n!$ for n vertices that arises from expanding $e^{-\lambda\mathbf{V}}$. Once the tree is known, one can integrate over the positions of vertices at a cost of $2^{-(n-1)(d+1)j}$. The number of possible sector assignments to the lines of the tree depends somewhat on the branching structure of the tree. It can be as large as $2^{-(n-1)(d-1)j}$, for example in the typical case of a linear tree. Combining this with the prefactor 2^{ldj} (see (III.1)) for all the lines we get $2^{(2-p)dj}$, independent of n. However it remains to bound the sum over possible "Wick contractions" for the loop lines.

In fermionic theories, this sum has the form of a determinant. To bound this determinant without expanding it fully (which would lead to the usual divergence of perturbation theory) one must exploit the Pauli exclusion principle. It states that there is, roughly speaking, only one $\bar{\psi}$ and one ψ per unit volume in phase space. The decay of the propagator in (III.1) damps contractions between ψ's and $\bar{\psi}$'s that are widely separated (in units of 2^{-j}) in position space. So we may pretend that there is only one ψ and one $\bar{\psi}$ in each sector of momentum space. (Technical details filling in this argument are given in [1b].) Since fields may only contract to other fields in the same sector (or neighbouring sectors), once sector attributions are made for each uncontracted field hooked to the spanning tree the pattern of Wick contractions is essentially determined.

It only remains to count the number of possible sector assignments to the ψ's and $\bar{\psi}$'s that are not in the spanning tree. At this point the key difference between two and three dimensions appears. Consider the typical case in which the spanning tree is a linear tree. Since sector indices have already been assigned to the lines of the spanning tree, we know the sector indices of two out of the four fields at each vertex. In $d = 2$, the factorized form of the vertex fixes the sector indices of the remaining two fields up to the logarithmic correction of the Lemma. In $d = 3$ there is no logarithm , but the "twist" costs an 2^{-j} per vertex. This means that a naive lower bound for the radius of convergence r_j is 2^j in three dimensions and $1/|\log j|$ in two dimensions.

We can actually get a better bound on r_j by using "anisotropic" sectors whose dimensions tangent to the Fermi surface are of order $2^{j/2}$ and whose remaining dimensions are still of order 2^j. Using Pauli's principle in corresponding anisotropic unit volume cells of phase space one proves that the radius of convergence r_j is at least $2^{j/2}$ in $d = 3$ and $O(1)$ in $d = 2$.

References

[1] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, a) “An Intrinsic $1/N$ Expansion for Many Fermion Systems”, to appear in EuroPhys. Lett.; b) Helv. Phys. Acta, **65** 679-721 (1992)

[2] J. Feldman and E. Trubowitz, Helv. Phys. Acta, **63** 156-260 (1990) and **64** 213-357 (1991).

[3] G. Benfatto and G. Gallavotti, Journ. Stat. Phys. **59**, 541-664 (1990)