Renormalization in Classical Mechanics and Many Body Quantum Field Theory

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We attempt to give a *pedagogical* introduction to perturbative renormalization. Our approach is to first describe, following Linstedt and Poincaré, the renormalization of formal perturbation expansions for quasi-periodic orbits in Hamiltonian mechanics. We then discuss, following [FT1,FT2], the renormalization of the formal ground state energy density of a many Fermion system. The construction of formal quasi-periodic orbits is carried out in detail to provide a relatively simple model for the considerably more involved, and perhaps less familiar, perturbative analysis of a field theory.

As we shall see, quasi-periodic orbits and many Fermion systems have a number of important features in common. In particular, as Poincaré observed in the classical case and [FT1,FT2] pointed out in the latter, the formal expansions considered here both contain divergent subseries.

I. Quasi-Periodic Orbits

Let $\mathbf{T}^d = \mathbf{R}^d/2\pi\mathbf{Z}^d$ be the d-dimensional torus and $B \subset \mathbf{R}^d$ a small ball centered at the origin. Fix $\omega \in \mathbf{R}^d$ and consider the Hamiltonian

$$< \omega, y > \tag{I.1}$$

on the phase space $\mathbf{T}^d \times B$. The corresponding equations of motion are

$$\begin{aligned} \frac{dx}{dt} &= \omega \\ \frac{dy}{dt} &= 0 \ . \end{aligned} \tag{I.2}$$

For each initial point $(x_0, \alpha) \in \mathbf{T}^d \times B$ there is a solution

$$(\omega t + x_0, \alpha) \ , \quad -\infty < t < \infty \qquad (I.3)_{(x_0,\alpha)}$$

of (I.2). We shall assume that the frequency vector ω satisfies the strongly nonresonant Diophantine condition

$$| < \omega, k > | \geq \frac{\text{const}}{|k|^\tau} \qquad \tau \ > \ d-1 \tag{I.4}$$

for all $k \neq 0$ in $\mathbf{Z}^d$. In this case, $(I.3)_{(x_0,\alpha)}$ winds densely around $\mathbf{T}^d \times \{\alpha\}$.

Consider the perturbed Hamiltonian

$$< \omega, y > +\varepsilon \left(P(x) + \frac{1}{2} < y, Q(x)y > +R(x,y) \right) \tag{I.5}$$

on $\mathbf{T}^d \times B$. Here, $P,\ Q = Q^T$ and R are real analytic, and

$$R(x,y) = O(|y|^3)\ . \tag{I.6}$$

The corresponding equations of motion are

$$\begin{aligned} \frac{dx}{dt} &= \omega + \varepsilon \left(Q(x)y + R_y \right) \\ \frac{dy}{dt} &= \quad -\varepsilon \left(P_x(x) + \frac{1}{2} < y, Q_x(x)y > +R_x \right) \end{aligned} \tag{I.7}$$

Recall that the composition

$$f(\omega t) = \sum_{k \in \mathbf{Z}^d} \hat{f}(k) e^{i<k,\omega>t}$$

where $f \in C^\infty\left(\mathbf{T}^d\right)$ is called a quasi-periodic function of t with frequency module

$$\left\{ < k, \omega > \mid k \in \mathbf{Z}^d \right\}.$$

The Fourier coefficient $\hat{f}(k),\ k \in \mathbf{Z}^d$, is defined by

$$\hat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} f(x) e^{-i<k,x>}\ . \tag{I.8}$$

One attempts to construct a formal, quasi-periodic solution to (I.7) of the form

$$(\omega t, 0) + (u(\omega t, \varepsilon), v(\omega t, \varepsilon)) \tag{I.9a}$$

$$\begin{aligned} u(x,\varepsilon) &\sim \sum_{j \geq 1} u_j(x) \varepsilon^j \\ v(x,\varepsilon) &\sim \sum_{j \geq 1} v_j(x) \varepsilon^j \end{aligned} \tag{I.9b}$$

where $u_j, v_j, j \geq 1$, are $\mathbf{R}^d$ valued real analytic functions on $\mathbf{T}^d$

Substituting, we obtain

$$\begin{aligned} \omega + \frac{d}{dt} u(\omega t) &= \omega + \varepsilon Q\left(\omega t + u(\omega t)\right) v(\omega t) + \varepsilon R_y\left(\omega t + u(\omega t), v(\omega t)\right) \\ \frac{d}{dt} v(\omega t) &= \quad -\varepsilon P_x\left(\omega t + u(\omega t)\right) - \frac{\varepsilon}{2} \left\langle v(\omega t), Q_x\left(\omega t + u(\omega t)\right) v(\omega t) \right\rangle \\ &\qquad - \varepsilon R_x\left(\omega t + u(\omega t), v(\omega t)\right) \end{aligned} \tag{I.10}$$

Since the trajectory ωt winds densely around $\mathbf{T}^d$, it follows that (I.9) is a solution of (I.7) if and only if $(u(x), v(x))$ is a solution of the first order, nonlinear partial differential equation

$$\begin{pmatrix} u_x\omega \\ v_x\omega \end{pmatrix} = \varepsilon I\,(x, u, v) \qquad (I.11a)$$

where

$$I\,(x, u, v) = \begin{pmatrix} Q(x+u)v + R_y(x+u, v) \\ -P_x(x+u) - \frac{1}{2} < v, Q_x(x+u)v > -R_x(x+u, v) \end{pmatrix} \qquad (I.11b)$$

and

$$u_x\omega \;=\; \begin{pmatrix} u^1_{x_1} & \dots & u^1_{x_d} \\ \vdots & \ddots & \vdots \\ u^d_{x_1} & \dots & u^d_{x_d} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_d \end{pmatrix} \;=\; \begin{pmatrix} \sum_{i=1}^d \; u^1_{x_i}\;\omega_i \\ \vdots \\ \sum_{i=1}^d \; u^d_{x_i}\;\omega_i \end{pmatrix}$$

is the directional derivative of u in the direction ω .

Matching powers of ε in (I.11), one finds

$$\begin{aligned} u_{1x}\omega &= 0 & \qquad u_{2x}\omega &= Q(x)v_1(x) \\ v_{1x}\omega &= -P_x(x) & \qquad v_{2x}\omega &= -P_{xx}(x)u_1(x) \end{aligned}$$

In general, if $u_i, v_i,\ i < j$ exist and are real analytic $\mathbf{R}^d$ valued functions on $\mathbf{T}^d$, then

$$u_{jx}\omega = r_j(x) \qquad\qquad v_{jx}\omega = s_j(x) \qquad (I.12j)$$

where $r_j, s_j,\ j \geq 1$, are also real analytic $\mathbf{R}^d$ valued functions on $\mathbf{T}^d$ recursively constructed from $u_i, v_i,\ i < j$ and P, Q, R.

Suppose $u_i, v_i,\ i < j$ exist and are real analytic. Then, formally solving (I.12j), one obtains

$$\begin{aligned} u_j(x) &= \sum_{k\in\mathbf{Z}^d} \frac{\hat{r}_j(k)}{i < k, \omega >} e^{i<k,x>} \\ v_j(x) &= \sum_{k\in\mathbf{Z}^d} \frac{\hat{s}_j(k)}{i < k, \omega >} e^{i<k,x>} \end{aligned} \qquad (I.13j)$$

By (I.4),

$$| - i < k, \omega >^{-1} | < \text{const } |k|^\tau$$

for $|k| > 0$. Observe that

$$|\hat{r}_j(k)|\ ,\ |\hat{s}_j(k)| < \text{const } e^{-\text{const}|k|}$$

since $r_j,\ s_j$ are real analytic on $\mathbf{T}^d$. Therefore, the series

$$\sum_{|k|>0} \frac{\hat{r}_j(k)}{i < k, \omega >} e^{i<k,x>}$$

$$\sum_{|k|>0} \frac{\hat{s}_j(k)}{i < k, \omega >} e^{i<k,x>}$$

converge absolutely to real analytic functions. In other words, the large momentum contribution to $(I.13j)$ is well defined.

On the other hand, the " free propagator" $-i < k, \omega >^{-1},\ k \in \mathbf{Z}^d$ has a "linear" singularity at $k = 0$. Consequently, $u_j,\ v_j$ are well defined if and only if $\hat{r}_j(0) = [r_j] = 0,\ \hat{s}_j(0) = [s_j] = 0$, where

$$[f] = \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} f(x)\ dx.$$

In particular, $u_1,\ v_1$ are well defined since $[P_x] = 0$. But, u_2 is undefined unless $[Q(x)v_1(x)] = 0$. We may say that each order of perturbation theory is "ultraviolet finite" , but potentially "infrared divergent". Ultraviolet refers to large momenta, or equivalently, small distances. Infrared refers to small momenta, or large distances.

The "infrared" divergent terms

$$\frac{[r_j]}{i < 0, \omega >} = \infty, \qquad \frac{[s_j]}{i < 0, \omega >} = \infty$$

can be removed by introducing counterterms $\rho_j,\ \sigma_j \in \mathbf{R}^d,\ \ j \geq 1$. Set

$$\begin{aligned} \rho(\varepsilon) &\sim \sum_{j\geq 1} \rho_j \varepsilon^j \\ \sigma(\varepsilon) &\sim \sum_{j\geq 1} \sigma_j \varepsilon^j \end{aligned} \tag{I.14}$$

and consider the modified equation

$$\begin{pmatrix} u_x \omega \\ v_x \omega \end{pmatrix} = \varepsilon I\,(x, u, v) - \begin{pmatrix} \rho \\ \sigma \end{pmatrix} \tag{I.15}$$

for $u, v, \rho,$ and σ . Equation (I.15) in contrast to (I.11) always has a unique (formal) solution u, v, ρ, σ with $[u] = 0$ and $[v] = 0$. In fact, proceeding by induction, if $u_i, v_i,\ \rho_i, \sigma_i,\ i < j$ exist and u_i, v_i are real analytic, then there are real analytic $\mathbf{R}^d$ valued functions r_j, s_j on

$\mathbf{T}^d$ and vectors $\rho_j,\ \sigma_j \in \mathbf{R}^d$ recursively constructed from $u_i, v_i, \rho_i, \sigma_i,\ i < j$ and P, Q, R such that

$$u_{jx}\omega = r_j(x) - \rho_j \qquad\qquad v_{jx}\omega = s_j(x) - \sigma_j \tag{I.16j}$$

It follows directly from the discussion of (I.12j) above that

$$\begin{aligned}
u(x,\varepsilon) &\sim \sum_{j\geq 1} \left(\sum_{|k|>0} \frac{\hat{r}_j(k)}{i < k, \omega >} e^{i<k,x>} \right) \varepsilon^j \\
v(x,\varepsilon) &\sim \sum_{j\geq 1} \left(\sum_{|k|>0} \frac{\hat{s}_j(k)}{i < k, \omega >} e^{i<k,x>} \right) \varepsilon^j \\
\rho(\varepsilon) &\sim \sum_{j\geq 1} \ [r_j]\, \varepsilon^j \\
\sigma(\varepsilon) &\sim \sum_{j\geq 1} \ [s_j]\, \varepsilon^j
\end{aligned} \tag{I.17}$$

is the unique, well defined formal power series solution of (I.15) with $[u] = 0$ and $[v] = 0$.

The remedy of the last paragraph for infrared divergence is on the face of it unacceptable, because a solution of (I.15) does not necessarily yield a quasi-periodic solution of (I.7) of the form (I.9). However, the free Hamiltonian $< \omega, y >$ generates the 2d-parameter family of orbits

$$(\omega t + x_0, \alpha)\ , \quad -\infty < t < \infty, \tag*{$(I.3)_{(x_0,\alpha)}$}$$

$(x_0, \alpha) \in \mathbf{T}^d \times B$. We have expanded around $(\omega t, 0),\ -\infty < t < \infty$ keeping the initial position x_0 and action α fixed at 0. We will now expand around $(I.3)_{(x_0,\alpha)}$ and renormalize the physical parameters x_0 and α to compensate for the additional term $-\begin{pmatrix} \rho \\ \sigma \end{pmatrix}$ in (I.15). That is, x_0 and α must be chosen to depend on ε in a specific way.

Reviewing the discussion leading up to (I.11) and then to (I.15) one sees that

$$(\omega t + x_0, \alpha) + (u(\omega t + x_0, \varepsilon), v(\omega t + x_0, \varepsilon)) \tag{I.18}$$

is a solution of (I.7) if and only if $(u(x), v(x))$ is a solution of

$$\begin{pmatrix} u_x\omega \\ v_x\omega \end{pmatrix} = \varepsilon I\,(x, u, v + \alpha) \tag{I.19}$$

and furthermore that

$$\begin{pmatrix} u_x\omega \\ v_x\omega \end{pmatrix} = \varepsilon I\,(x, u, v + \alpha) - \begin{pmatrix} \rho \\ \sigma \end{pmatrix} \tag{I.20}$$

has a unique formal power series solution (u, v, ρ, σ),

$$\begin{aligned}
u(x, \alpha, \varepsilon) &\sim \sum_{j\geq 1} \left(\sum_{|k|>0} \frac{\hat{r}_j(k, \alpha)}{i < k, \omega >} e^{i<k,x>} \right) \varepsilon^j \\
v(x, \alpha, \varepsilon) &\sim \sum_{j\geq 1} \left(\sum_{|k|>0} \frac{\hat{s}_j(k, \alpha)}{i < k, \omega >} e^{i<k,x>} \right) \varepsilon^j \\
\rho(\alpha, \varepsilon) &\sim \sum_{j\geq 1} \left[r_j(\alpha) \right] \varepsilon^j \\
\sigma(\alpha, \varepsilon) &\sim \sum_{j\geq 1} \left[s_j(\alpha) \right] \varepsilon^j
\end{aligned} \qquad (I.21)_\alpha$$

with $[u] = 0, [v] = 0$, depending real analytically on $\alpha \in B$. Observe that x_0 does not appear in (I.19) or (I.20). By definition, (I.11) is *renormalizable* if it is possible to solve for α as a function (formal power series) of ε so that

$$\rho\left(\alpha(\varepsilon), \varepsilon\right) = 0 \quad \sigma\left(\alpha(\varepsilon), \varepsilon\right) = 0. \qquad (I.22)$$

Clearly,

$$(\omega t + x_0, \alpha(\varepsilon)) + (u(\omega t + x_0, \alpha(\varepsilon), \varepsilon), v(\omega t + x_0, \alpha(\varepsilon), \varepsilon))$$

is a formal quasi-periodic solution to (I.7) when (I.11) is renormalizable.

Matching powers of ε in (I.20), one finds

$$u_{1x}\omega = r_1(x, \alpha) - \rho_1 = Q(x)\alpha + R_y(x, \alpha) - \rho_1 \ .$$

As before, $[u_{1x}\omega] = 0$ implies $\rho_1 = [Q]\alpha + O\left(\alpha^3\right)$ with the result that

$$\rho = \varepsilon \left[Q\right] \alpha \ + \ O\left(\varepsilon^2\right) + O\left(\alpha^3\right) \ .$$

Suppose

$$\det\left[Q\right] \neq 0 \ . \qquad (I.23)$$

Then, by the formal implicit function theorem, there is a unique formal power series

$$\alpha(\varepsilon) \sim \sum_{j\geq 1} \alpha_j \varepsilon^j \qquad (I.24)$$

such that $\rho\left(\alpha(\varepsilon),\varepsilon\right)=0$. What about $\sigma\left(\alpha(\varepsilon),\varepsilon\right)$?

There are 2d counterterms $\rho,\ \sigma$ but only d parameters α. At first sight, one would expect that it is not possible to choose α (depending on ε) so that ρ, and σ are both 0. However, Poincaré showed that $\sigma\left(\alpha(\varepsilon),\varepsilon\right)$ automatically vanishes when $\alpha(\varepsilon)$ is the formal power series (I.24). The idea is to conjugate (I.5) by a formal symplectic map to a Hamiltonian

$$< \omega, \alpha + y > + g(y,\alpha,\varepsilon) \tag{I.25}$$

that only depends on y. The counterterm σ generated by (I.25) is clearly zero, since

$$I\left(x,u,v\right)=\begin{pmatrix} g(v,\alpha,\varepsilon) \\ 0 \end{pmatrix} .$$

Recall that

$$F\ :\ (x,y)\longrightarrow(x+f_1(x,y,\varepsilon),\ y+f_2(x,y,\varepsilon))$$

$$f_i(x,y,\varepsilon)=\sum_{j\geq1} f_{i,j}(x,y)\varepsilon^j\ ,\quad i=1,2$$

is a formal symplectic map on $\mathbf{T}^d\times B$ when $f_{i,j}$ is real analytic on $\mathbf{T}^d\times B_j$, B_j a small ball centered at the origin in $\mathbf{R}^d$, and the $2d\times 2d$ Jacobian satisfies

$$\begin{pmatrix} \mathbf{1}+f_{1x} & f_{1y} \\ f_{2x} & \mathbf{1}+f_{2y} \end{pmatrix}\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{1}+f_{1x} & f_{1y} \\ f_{2x} & \mathbf{1}+f_{2y} \end{pmatrix}^T=\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

order by order in ε . Poincaré ([P], [A1]) showed that there is a unique formal symplectic map F with $[f_i(\cdot,y,\alpha)]=0$, i=1,2, depending analytically on $\alpha\in B$ and a unique formal power series

$$g(y,\alpha,\varepsilon)=\sum_{j\geq1} g_j(y,\alpha)\varepsilon^j$$

also depending analytically on α such that

$$\begin{aligned} < \omega,\alpha+y+f_2 > +\varepsilon P(x+f_1)+\frac{\varepsilon}{2} < (\alpha+y+f_2),Q(x+f_1)(\alpha+y+f_2) > \\ +\varepsilon R(x+f_1,\alpha+y+f_2) \\ = < \omega,\alpha+y > +g(y,\alpha,\varepsilon). \end{aligned} \tag{I.26}$$

It follows from (I.26) that a formal quasi-periodic orbit for $< \omega,\alpha+y > +g(y,\alpha,\varepsilon)$ can be mapped by F to a formal quasi-periodic orbit for (I.5). But, as remarked above,

σ disappears when the Hamiltonian depends on y alone. The construction of solutions (I.18) may be completed in this way. Nevertheless, it is instructive to directly verify that σ vanishes.

Differentiate (I.26) and set $y = 0$ to obtain

$$\begin{pmatrix} f_{2x}{}^T \omega \\ f_{2y}{}^T \omega \end{pmatrix} + \varepsilon \begin{pmatrix} \mathbf{1} + f_{1x}{}^T & f_{2x}{}^T \\ f_{1y}{}^T & \mathbf{1} + f_{2y}{}^T \end{pmatrix} \begin{pmatrix} \nabla_x \\ \nabla_y \end{pmatrix} \Big|_{(x+f_1, \alpha+f_2)} \phi - \begin{pmatrix} 0 \\ g_y(0, \varepsilon) \end{pmatrix} = 0 \qquad (I.27)$$

where

$$\phi = P(x) + \frac{1}{2} < y, Q(x)y > + R(x, y, \varepsilon)$$

and

$$f_{2x}{}^T \omega = \begin{pmatrix} f^1_{2x_1} & \cdots & f^d_{2x_1} \\ \vdots & \ddots & \vdots \\ f^1_{2x_d} & \cdots & f^d_{x_d} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_d \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^d f^i_{2x_1} \ \omega_i \\ \vdots \\ \sum_{i=1}^d f^i_{x_d} \ \omega_i \end{pmatrix}$$

Multiplication on the left by

$$\begin{pmatrix} \mathbf{1} + f_{1x} & f_{1y} \\ f_{2x} & \mathbf{1} + f_{2y} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} -f_{1y} & \mathbf{1} + f_{1x} \\ -\mathbf{1} - f_{2y} & f_{2x} \end{pmatrix}$$

yields

$$\begin{pmatrix} f_{1x}(x,0)\omega \\ f_{2x}(x,0)\omega \end{pmatrix} = \varepsilon I \left(x, f_1(x,0), \alpha + f_2(x,0)\right) - \begin{pmatrix} (\mathbf{1} + f_{1x}(x,0))g_y(0) \\ f_{2x}(x,0)g_y(0) \end{pmatrix} \qquad (I.28)$$

because

$$\begin{pmatrix} -f_{1y} & \mathbf{1} + f_{1x} \\ -\mathbf{1} - f_{2y} & f_{2x} \end{pmatrix} \begin{pmatrix} f_{2x}{}^T \omega \\ f_{2y}{}^T \omega \end{pmatrix} = \begin{pmatrix} (-f_{1y} f_{2x}{}^T + (\mathbf{1} + f_{1x}) f_{2y}{}^T)\omega \\ ((-\mathbf{1} - f_{2y}) f_{2x}{}^T + f_{2x} f_{2y}{}^T)\omega \end{pmatrix}$$
$$= \begin{pmatrix} -f_{1x}(x,0)\omega \\ -f_{2x}(x,0)\omega \end{pmatrix}$$

since F is symplectic.

At order ε, (I.28) implies $g_{1y}(0) = [Q]\alpha + O(\alpha^2)$, since

$$f_{1x}\omega = Q(x)\alpha + R_y(x, \alpha) - g_{1y}(0) \ .$$

Once again condition (I.23) and the formal implicit function theorem yield a unique formal power series $\alpha(\varepsilon)$ with $g_y(0, \alpha(\varepsilon), \varepsilon) = 0$. It follows that for this choice of α

$$\Big(f_1(x,0), \ f_2(x,0), \ 0, \ 0\Big) \qquad (I.29)$$

is the unique solution $(I.21)_\alpha$, (u, v, ρ, σ), of (I.20) with $[u] = [f_1(x,0)] = 0, \ [v] = [f_2(x,0)] = 0$ since

$$\rho(\alpha(\varepsilon), \varepsilon) = (\mathbf{1} + f_{1x}(x,0))g_y(0) = 0 \quad and \quad \sigma(\alpha(\varepsilon), \varepsilon) = f_{2x}(x,0)g_y(0) = 0.$$

Therefore, condition (I.23) implies that (I.11) is renormalizable.

We have

<u>Theorem 1</u> (Linstedt, Poincaré)
Let $P(x)$ and $R(x,y) = O\left(|y|^3\right)$ be real analytic functions on $\mathbf{T}^d$ and $\mathbf{T}^d \times B$. Let $Q = Q^T$ be a real analytic, $d \times d$ symmetric matrix valued function on $\mathbf{T}^d$ satisfying

$$\det\left(\int_{\mathbf{T}^d} Q_{ij}(x)\ dx\right) \neq 0\ .$$

Suppose, $\omega \in \mathbf{R}^d$ satisfies the Diophantine condition

$$|<\omega, k>| \geq \frac{\text{const}}{|k|^\tau} \qquad \tau\ >\ d-1$$

for all $k \neq 0$ in $\mathbf{Z}^d$. Then, for each x_0 in $\mathbf{T}^d$

$$(\omega t + x_0, \alpha(\varepsilon)) + (u(\omega t + x_0, \alpha(\varepsilon), \varepsilon), v(\omega t + x_0, \alpha(\varepsilon), \varepsilon))$$

$$\begin{aligned} u(x) &\sim \sum_{j\geq 1} u_j(x,\alpha)\varepsilon^j \\ v(x) &\sim \sum_{j\geq 1} v_j(x,\alpha)\varepsilon^j \\ \alpha(\varepsilon) &\sim \sum_{j\geq 1} \alpha_j\varepsilon^j \end{aligned}$$

is a formal quasi-periodic solution of the Hamiltonian system

$$\begin{aligned} \frac{dx}{dt} &= \omega + \varepsilon\left(Q(x)y + R_y\right) \\ \frac{dy}{dt} &= \quad -\varepsilon\left(P_x(x) + \frac{1}{2} < y, Q_x(x)y > + R_x\right) \end{aligned}$$

where $u_j, v_j, j \geq 1$, are the $\mathbf{R}^d$ valued real analytic functions on $\mathbf{T}^d \times B$ constructed in $(I.21)_\alpha$ and $\alpha(\varepsilon)$ is given by (I.24).

To make Theorem 1 more concrete we elaborate on the form of $u_j, v_j, j \geq 1$. To do this, set

$$z\ =\begin{pmatrix} u \\ v \end{pmatrix} \qquad \kappa\ =\begin{pmatrix} \rho \\ \sigma \end{pmatrix}$$

and

$$\mathcal{I}\,(x,z) = \begin{pmatrix} Q(x+u)(v+\alpha) + R_y(x+u, v+\alpha) \\ -P_x(x+u) - \frac{1}{2} < (v+\alpha), Q_x(x+u)(v+\alpha) > -R_x(x+u, v+\alpha) \end{pmatrix} \qquad (I.30)$$

Expanding,

$$\mathcal{I}\,(x,z) \;=\; \sum_{n\,\geq\,0} \; \mathcal{I}_n(x)(z) \qquad (I.31)$$

where

$$\mathcal{I}_n(x)(y_1,\cdots,y_n) \;=\; \frac{1}{n!}\;\frac{\partial^n}{\partial t_1\;...\;\partial t_n}\mathcal{I}\left(x, \sum_{i=1}^{n}\; t_i y_i\right)\;\Big|_{t_1=\cdots=t_n=0}$$

for $y_i \in \mathbf{R}^{2d}$, $i = 1,\cdots,n$, and

$$\mathcal{I}_n(x)(z) \;=\; \mathcal{I}_n(x)(z,\;\cdots,\;z)$$

for all $n \;\geq\; 0$.

Inverting the directional derivative $< \omega, \nabla_x >$ on the complement of the constant functions, (I.20) may be rewritten in the form

$$\begin{aligned} z(x) \;&= < \omega, \nabla_x >^{-1}\,(\varepsilon\mathcal{I}\,(x, z(x)) - \kappa) \\ \kappa \;&=\; [\varepsilon\mathcal{I}\,(\cdot, z)] \\ &=\; \frac{1}{(2\pi)^d}\int_{\mathbf{T}^d}\; \varepsilon\mathcal{I}\,(x, z(x))\;\,dx\;. \end{aligned} \qquad (I.32)$$

Substituting the right hand side of (I.31), the last equation becomes

$$\begin{aligned} z(x) \;&=\; \sum_{n\,\geq\,0}\; < \omega, \nabla_x >^{-1}\,(\varepsilon\mathcal{I}_n(x)(z(x)) - [\varepsilon\mathcal{I}_n(\cdot)(z)]) \\ \kappa \;&=\; \sum_{n\,\geq\,0}\; [\varepsilon\mathcal{I}_n(\cdot)(z)]\;. \end{aligned} \qquad (I.33)$$

Using the multilinearity of $\mathcal{I}_n(x)(y_1,\cdots,y_n)$ one sees that the Fourier coefficient

$$\frac{1}{(2\pi)^d}\;\int_{\mathbf{T}^d}\;\mathcal{I}_n(x)(z(x))\;e^{-i<k,x>}\;d^dx$$

$$= \frac{1}{(2\pi)^d}\;\int_{\mathbf{T}^d}\;\mathcal{I}_n(x)\left(\sum_{k_1\in\mathbf{Z}^d}\;\hat{z}(k_1)\;e^{i<k_1,x>},\;\cdots,\;\sum_{k_n\in\mathbf{Z}^d}\;\hat{z}(k_n)\;e^{i<k_n,x>}\right)\;e^{-i<k,x>}\;d^dx$$

$$
\begin{aligned}
&= \sum_{k_1,\cdots,k_n\in\mathbf{Z}^d} \frac{1}{(2\pi)^d}\int_{\mathbf{T}^d} \mathcal{I}_n(x)(\hat z(k_1),\cdots,\hat z(k_n))\; e^{i<k_1+\cdots+k_n-k,x>}\; d^dx \\
&= \sum_{k_1,\cdots,k_n\in\mathbf{Z}^d} \widehat{\mathcal{I}}_n(k-k_1-\cdots-k_n)(\hat z(k_1),\cdots,\hat z(k_n))
\end{aligned} \tag{I.34}
$$

where

$$
\widehat{\mathcal{I}}_n(k)(y_1,\cdots,y_n) \;=\; \frac{1}{(2\pi)^d}\int_{\mathbf{T}^d} \mathcal{I}_n(x)(y_1,\cdots,y_n)\; e^{-i<k,x>}\; d^dx\ .
$$

Observe that $\widehat{\mathcal{I}}_n(k)(y_1,\cdots,y_n)$, $n\ge 0$, is an $\mathbf{R}^{2d}$ valued multilinear form on $\left(\mathbf{R^{2d}}\right)^n$ and, in particular, that $\widehat{\mathcal{I}}_0(k)$ is a vector. Combining (I.33) and (I.34), we obtain

$$
\begin{aligned}
z(x) \;&=\; \varepsilon \sum_{n\,\ge\,0}\; \sum_{\substack{k\in\mathbf{Z}^d\\ k\neq 0}}\; \sum_{k_1,\cdots,k_n\in\mathbf{Z}^d} \frac{e^{i<k,x>}}{i<k,\omega>}\, \widehat{\mathcal{I}}_n(k-k_1-\cdots-k_n)(\hat z(k_1),\cdots,\hat z(k_n)) \\
&= \varepsilon \sum_{n\,\ge\,0}\; \sum_{\substack{k_1,\cdots,k_{n+1}\in\mathbf{Z}^d\\ k_1+\cdots+k_{n+1}\neq 0}} \frac{1}{i<k_1+\cdots+k_{n+1},\omega>} \\
&\qquad\qquad \times \widehat{\mathcal{I}}_n(k_{n+1})(\hat z(k_1),\cdots,\hat z(k_n))\; e^{i<k_1+\cdots+k_{n+1},x>}
\end{aligned} \tag{I.35}
$$

and

$$
\kappa \;=\; \varepsilon \sum_{n\,\ge\,0}\; \sum_{k_1,\cdots,k_n\in\mathbf{Z}^d} \widehat{\mathcal{I}}_n(-k_1-\cdots-k_n)(\hat z(k_1),\cdots,\hat z(k_n))\ . \tag{I.36}
$$

The formal power series solution to (I.20) is generated by iterating (I.35) starting with $z=0$. The first iterate is

$$
\varepsilon \sum_{\substack{k_1\in\mathbf{Z}^d\\ k_1\neq 0}} \frac{1}{i<k_1,\omega>}\, \widehat{\mathcal{I}}_0(k_1)\; e^{i<k_1,x>}\ .
$$

The second

$$
\begin{aligned}
&\sum_{n\,\ge\,0} \varepsilon^{n+1} \sum_{\substack{k_1,\cdots,k_n\in\mathbf{Z}^d\\ k_1\neq 0\cdots k_n\neq 0}}\; \sum_{\substack{k_{n+1}\in\mathbf{Z}^d\\ k_1+\cdots+k_{n+1}\neq 0}} \frac{1}{i<k_1+\cdots+k_{n+1},\omega>} \prod_{i=1}^{n} \frac{1}{i<k_i,\omega>} \\
&\qquad\qquad \times \widehat{\mathcal{I}}_n(k_{n+1})(\widehat{\mathcal{I}}_0(k_1),\cdots,\widehat{\mathcal{I}}_0(k_n))\; e^{i<k_1+\cdots+k_{n+1},x>}\ .
\end{aligned}
$$

At the same time, (I.36) yields the expansion for κ .

As a simple example, observe that for every triple $m_1, m_2, m_3 \in \mathbf{Z}^d$ satisfying

$$m_1 \neq 0 \qquad m_1 + m_2 \neq 0 \qquad m_1 + m_2 + m_3 \neq 0$$

repeated iteration of (I.35) produces the term

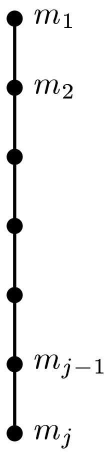
$$\prod_{s=1}^{3} \frac{1}{i < \sum_{t \le s} m_t, \omega >} \ \widehat{\mathcal{I}}_1(m_3) \left(\widehat{\mathcal{I}}_1(m_2)(\widehat{\mathcal{I}}_0(m_1)) \right) \ e^{i<m_1+m_2+m_3,x>} \tag{I.37}$$

of order ε^3 . A more complicated example of order ε^8 is the product of the rows

$$\begin{gathered}
\frac{1}{i<m_1,\omega>} \ \frac{1}{i<m_2,\omega>} \ \frac{1}{i<m_4,\omega>} \ \frac{1}{i<m_5,\omega>} \ \frac{1}{i<m_6,\omega>} \\
\frac{1}{i<m_1+m_2+m_3,\omega>} \ \frac{1}{i<m_4+m_5+m_6+m_7,\omega>} \\
\frac{1}{i<m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8,\omega>} \\
\widehat{\mathcal{I}}_2(m_8) \left(\widehat{\mathcal{I}}_2(m_3) \left(\widehat{\mathcal{I}}_0(m_1), \widehat{\mathcal{I}}_0(m_2) \right), \widehat{\mathcal{I}}_3(m_7) \left(\widehat{\mathcal{I}}_0(m_4), \widehat{\mathcal{I}}_0(m_5), \widehat{\mathcal{I}}_0(m_6) \right) \right) \ . \\
e^{i<m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8,x>}
\end{gathered} \tag{I.38}$$

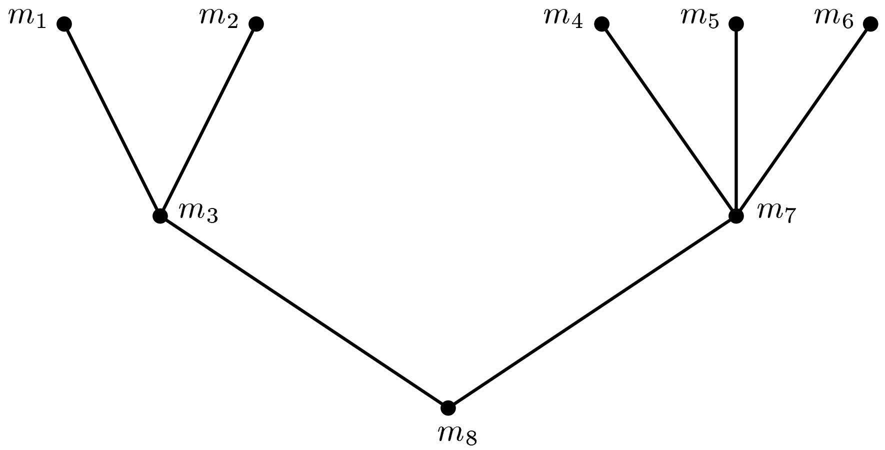
Here, the momenta $m_s,\ s = 1, \cdots, 8$ are chosen so that none of the denominators vanish. These examples exhibit a tree structure typical of iterative constructions. This structure is most conveniently exploited by introducing the following notation.

Let $\mathcal{T}_j$ be the set of all rooted planar trees with nodes $\nu = 1,\ \cdots,\ j$ and root j . Recall that a tree is a graph without loops and a rooted tree is a tree with a distinguished node. For example,



$(I.39j)$

and



$$(I.40)$$

There are $\frac{1}{j}\begin{pmatrix} 2j-2 \\ j-1 \end{pmatrix} \le 4^j$ rooted planar trees of order j . (See, [GJ] P.112)

We now inductively associate a vector $\mathcal{C}_T(m_1,\cdots,m_j) \in \mathbf{R}^{2d}$ to each tree $T \in \mathcal{T}_j$ and assignment of momenta $m_\nu \in \mathbf{Z}^d$ to the nodes $\nu,\ v = 1,\cdots,j$ of T by composing the multilinear forms $\widehat{\mathcal{I}}_n(k)(y_1,\cdots,y_n),\ n \ge 0,\ k \in \mathbf{Z}^d$. Suppose, $T = \cdot$ is the unique tree in $\mathcal{T}_1$, then

$$\mathcal{C}_T(m_1) \ = \ \widehat{\mathcal{I}}_0(m_1) \ . \tag{I.41}$$

Suppose, $T \in \mathcal{T}_j,\ j > 1$. Let $b \ge 1$ be the number of branches issuing from the root node of T . Removing the root of T , we obtain b trees $T_1,\ \cdots\ ,T_b$ of orders $j_1,\ \cdots\ ,j_b$ with $j_1 + \cdots + j_b = j-1$. Then,

$$\mathcal{C}_T(m_1,\cdots,m_j) \ =$$

$$\widehat{\mathcal{I}}_b(m_j)\left(\mathcal{C}_{T_1}(m_1,\cdots,m_{j_1}),\mathcal{C}_{T_2}(m_{j_1+1},\cdots,m_{j_1+j_2}),\ \cdots,\ \mathcal{C}_{T_b}(m_{j_1+\cdots+j_{b-1}+1},\cdots,m_{j-1})\right) \tag{I.42}$$

For example

$$\mathcal{C}_{(I.39j)}(m_1,\cdots,m_j) \ = \ \widehat{\mathcal{I}}_1(m_j)\Big(\ \cdots\ \widehat{\mathcal{I}}_1(m_3)\left(\widehat{\mathcal{I}}_1(m_2)(\widehat{\mathcal{I}}_0(m_1))\right)\Big)$$

and

$$\mathcal{C}_{(I.40)}(m_1,\cdots,m_8) \ =$$

$$\widehat{\mathcal{I}}_2(m_8)\left(\widehat{\mathcal{I}}_2(m_3)\left(\widehat{\mathcal{I}}_0(m_1),\widehat{\mathcal{I}}_0(m_2)\right),\widehat{\mathcal{I}}_3(m_7)\left(\widehat{\mathcal{I}}_0(m_4),\widehat{\mathcal{I}}_0(m_5),\widehat{\mathcal{I}}_0(m_6)\right)\right) \ .$$

Finally, the value $\mathrm{val}(T)(m_1,\cdots,m_j)$ of a tree $T\in\mathcal{T}_j$ with momenta m_ν, $\nu=1,\cdots,j$ assigned to the nodes of T is defined by

$$\mathrm{val}(T)(m_1,\cdots,m_j) \;=\; \prod_{\text{nodes }\nu\text{ in }T}{}^{\prime}\ \frac{1}{i<\sum_{\nu'\succeq\nu}\ m_{\nu'}\ ,\omega>}\ \ \mathcal{C}_T(m_1,\cdots,m_j) \qquad (I.43)$$

Here,

$$\prod_{\text{nodes }\nu\text{ in }T}{}^{\prime}\ \frac{1}{d_\nu} \;=\; \begin{cases}\prod_{\text{nodes }\nu\text{ in }T}\frac{1}{d_\nu} & d_\nu\neq 0,\ \nu=1,\cdots,j\\ 0 & \text{otherwise}\end{cases}$$

Also, $\nu'\succeq\nu$ if there is a path in T from ν to ν' that is directed away from the root. In other words, ν' is at or above ν in the partial order induced by the rooted tree structure. For example,

$$\mathrm{val}(1.39j)(m_1,\cdots,m_j) \;=\; \prod_{1\le s\le j}{}^{\prime}\ \frac{1}{i<m_1+\cdots+m_s,\omega>}\ \ \mathcal{C}_{(I.39j)}(m_1,\cdots,m_j) \qquad (I.44)$$

and

$$\mathrm{val}(1.40)(m_1,\cdots,m_8) \;=$$

$$\frac{1}{i<m_1,\omega>}\,\frac{1}{i<m_2,\omega>}\,\frac{1}{i<m_4,\omega>}\,\frac{1}{i<m_5,\omega>}\,\frac{1}{i<m_6,\omega>}$$
$$\frac{1}{i<m_1+m_2+m_3,\omega>}\,\frac{1}{i<m_4+m_5+m_6+m_7,\omega>}$$
$$\frac{1}{i<m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8,\omega>}$$
$$\mathcal{C}_{(I.40)}(m_1,\cdots,m_8)\quad .$$

We shall need a small modification of $\mathrm{val}(T)(m_1,\cdots,m_j)$. Set

$$\mathrm{val}(T)_0(m_1,\cdots,m_j) \;=\; \prod_{\text{nodes }\nu\neq\text{ root}}{}^{\prime}\ \frac{1}{i<\sum_{\nu'\succeq\nu}\ m_{\nu'}\ ,\omega>}\ \ \mathcal{C}_T(m_1,\cdots,m_j) \qquad (I.45)$$

for $m_1+\cdots+m_j=0$.

Assembling all of this notation, one can easily verify

Lemma I.1 (Tree Expansion)

Let $P(x)$ and $R(x,y) = O\left(|y|^3\right)$ be real analytic functions on $\mathbf{T}^d$ and $\mathbf{T}^d \times B$. Let $Q = Q^T$ be a real analytic, $d \times d$ symmetric matrix valued function on $\mathbf{T}^d$ and $\alpha \in B$ Suppose, $\omega \in \mathbf{R}^d$ satisfies the Diophantine condition

$$| < \omega, k > | \geq \frac{\text{const}}{|k|^\tau} \qquad \tau \ > \ d-1$$

for all $k \neq 0$ in $\mathbf{Z}^d$. Then the unique formal power series solution

$$u(x) \sim \sum_{j \geq 1} \ u_j(x,\alpha)\varepsilon^j$$
$$v(x) \sim \sum_{j \geq 1} \ v_j(x,\alpha)\varepsilon^j$$
$$\rho(\varepsilon) \sim \sum_{j \geq 1} \ \rho_j \varepsilon^j$$
$$\sigma(\varepsilon) \sim \sum_{j \geq 1} \ \sigma_j \varepsilon^j$$

to

$$\begin{pmatrix} u_x \omega \\ v_x \omega \end{pmatrix} = \varepsilon I\left(x, u, v+\alpha, \varepsilon\right) - \begin{pmatrix} \rho \\ \sigma \end{pmatrix} \tag{I.20}$$

with $[u] = 0,\ [v] = 0$ is given by

$$\begin{pmatrix} u_j(x) \\ v_j(x) \end{pmatrix} = \sum_{m_1,\cdots,m_j \in \mathbf{Z}^d} \ \sum_{T \in \mathcal{T}_j} \ \mathrm{val}(T)(m_1,\cdots,m_j)\ e^{i<m_1+\cdots+m_j, x>}$$
$$\begin{pmatrix} \rho_j \\ \sigma_j \end{pmatrix} = \sum_{\substack{m_1,\cdots,m_j \in \mathbf{Z}^d \\ m_1+\cdots+m_j=0}} \ \sum_{T \in \mathcal{T}_j} \ \mathrm{val}(T)_0(m_1,\cdots,m_j)$$

How large is $\mathrm{val}(T)(m_1,\cdots,m_j)$? To begin with,

$$|\widehat{\mathcal{I}}_n(k)(y_1,\cdots,y_n)| \ \leq \ \text{const}^n \ e^{-\text{const}|k|} \ |y_1|\cdots|y_n|$$

for all $n \ \geq \ 0,\ k \in \mathbf{Z}^d$ and $y_1,\cdots,y_n \in \mathbf{R}^{2d}$. This bound is obtained by applying Cauchy's theorem in a complex neighborhood of $\mathbf{T}^d \times \{0\}$. It follows directly that,

$$|\mathcal{C}_T(m_1,\cdots,m_j)| \ \leq \ \text{const}^j \ e^{-\text{const}(|m_1|+\cdots+|m_j|)} \tag{I.46}$$

One now has to estimate

$$\Big| \prod_{\text{nodes } \nu \text{ in } T}' \ \frac{1}{i < \sum_{\nu' \succeq \nu} \ m_{\nu'} \ , \omega >} \Big| \ .$$

The crudest estimate using (I.4) is

$$\Big| \prod_{\text{nodes } \nu \text{ in } T}' \frac{1}{i < \sum_{\nu' \succeq \nu} m_{\nu'} , \omega >} \Big| \le \prod_{\text{nodes } \nu \text{ in } T} \text{const} \left(\sum_{\nu' \succeq \nu} |m_{\nu'}| \right)^{\tau}$$
$$\le \text{const}^j \left(\sum_{\nu=1}^{j} |m_\nu| \right)^{\tau j} .$$

A much more subtle bound is

Theorem I.2 (Siegel's Third Lemma)

Suppose $\omega \in \mathbf{R}^d$ satisfies

$$| < \omega, k > | \ge \frac{\text{const}}{|k|^\tau} \qquad \tau > d-1 \tag{I.4}$$

for all $k \ne 0$ in $\mathbf{Z}^d$. Let $T \in \mathcal{T}_j$. If $m_\nu \in \mathbf{Z}^d,\ \nu = 1, \cdots, j$ is an assignment of momenta to the nodes of T satisfying

$$\sum_{\mu' \succeq \mu} m_{\mu'} \ne \sum_{\nu' \succeq \nu} m_{\nu'}$$

whenever μ lies strictly above ν on T, then

$$\Big| \prod_{\text{nodes } \nu \text{ in } T}' \frac{1}{i < \sum_{\nu' \succeq \nu} m_{\nu'} , \omega >} \Big| \le \text{const}^j \frac{\left(\prod_{m_\nu \ne 0} |m_\nu| \right)^{3\tau}}{\left(\sum_\nu |m_\nu| \right)^{2\tau}} .$$

Proof: See [Si], [E$_1$] and [E$_2$] .

If Theorem I.2 held for every T and assignment of momenta, then the tree expansion would converge absolutely for small ε since

$$\sum_{j \ge 1} \varepsilon^j \sum_{m_1, \cdots, m_j \in \mathbf{Z}^d} \sum_{T \in \mathcal{T}_j} \left| \text{val}(T)(m_1, \cdots, m_j)\ e^{i < m_1 + \cdots + m_j, x >} \right|$$
$$< \sum_{j \ge 1} \varepsilon^j \sum_{m_1, \cdots, m_j \in \mathbf{Z}^d} \sum_{T \in \mathcal{T}_j} \left| \text{val}(T)(m_1, \cdots, m_j) \right|$$
$$< \sum_{j \ge 1} \varepsilon^j\ 4^j \sum_{m_1, \cdots, m_j \in \mathbf{Z}^d} \text{const}^j \left(\prod_{m_\nu \ne 0} |m_\nu| \right)^{3\tau} e^{-\text{const}(|m_1| + \cdots + |m_j|)}$$
$$< \infty \quad \left(|\mathcal{T}_j| < 4^j \text{ is used in the preceding line} \right) .$$

However, as Poincaré observed, this assumption is too naive, because for some trees and momenta assignments that violate the hypothesis of Theorem I.2 $\mathrm{val}(T)(m_1, \cdots, m_j)$ is anomolously large.

By Dirichlet's theorem (See [Sc], P.27), there are infinitely many vectors $m^{(n)}$ in $\mathbf{Z}^d$ $n \geq 1$ satisfying

$$|< m^{(n)}, \omega >| \ \leq \ \text{const } j_n^{-(d-1)}$$

where

$$j_n \ = \ \max \left\{ |m_i^{(n)}| \ \ 1 \leq i \leq d-1 \right\}$$

To illustrate the phenomenon in its simplest form, suppose that for all $n \geq 1$

$$|\widehat{\mathcal{I}}_0(m^{(n)})| \ \geq \ \text{const } e^{-\text{const} j_n}.$$

and $\widehat{\mathcal{I}}_1(\mathbf{1})$, $\widehat{\mathcal{I}}_1(-\mathbf{1}) \neq 0$, where $\mathbf{1} = (1, 0, \cdots, 0)$. Then, in case j_n is even,

$$\mathrm{val}(1.39 j_n)(m^{(n)}, \mathbf{1}, -\mathbf{1}, \cdots, -\mathbf{1}, \mathbf{1}) \ =$$

$$\left(\frac{1}{i < m^{(n)}, \omega >} \right)^{\frac{j_n}{2}} \ \left(\frac{1}{i < m^{(n)} + \mathbf{1}, \omega >} \right)^{\frac{j_n}{2}} \ \mathcal{C}_{(I.39 j_n)}(m^{(n)}, \mathbf{1}, -\mathbf{1}, \cdots, -\mathbf{1}, \mathbf{1}) \qquad (I.47)$$

brutally violating the hypothsis of Theorem I.2, and

$$\Big| \ \mathrm{val}(1.39 j_n)(m^{(n)}, \mathbf{1}, -\mathbf{1}, \cdots, -\mathbf{1}, \mathbf{1}) \ \Big| \ \ \geq \ \ \text{const}^{j_n} \ \left(j_n^{\frac{d-1}{2}} \right)^{j_n} \ e^{-\text{const} j_n} \ .$$

The case of odd j_n is similar. By Lemma I.1 and the last estimate

$$\sum_{n \geq 1} \varepsilon^{j_n} \ \mathrm{val}(1.39 j_n)(m^{(n)}, \mathbf{1}, -\mathbf{1}, \cdots, -\mathbf{1}, \mathbf{1}) \ e^{i < m^{(n)} + \mathbf{1}, x >} \qquad (I.48)$$

is a subseries of the tree expansion that diverges for all $\varepsilon \neq 0$.

Kolmogorov and Arnold ([K],[A2]) proved that the formal solutions of Theorem I.1 converge for small ε. That is,

$$\sum_{j \geq 1} \varepsilon^j \ \Big| \ \sum_{m_1, \cdots, m_j \in \mathbf{Z}^d} \ \sum_{T \in \mathcal{T}_j} \ \mathrm{val}(T)(m_1, \cdots, m_j) \ e^{i < m_1 + \cdots + m_j, x >} \ \Big|$$

converges, even though (as shown above),

$$\sum_{j\geq 1} \varepsilon^j \sum_{m_1,\cdots,m_j\in\mathbf{Z}^d} \sum_{T\in\mathcal{T}_j} \Big|\mathrm{val}(T)(m_1,\cdots,m_j)\Big|$$

diverges. Their proof is indirect. They essentially solve (I.20) by Newton's method. The super-convergence of the iterates is used to control the small divisors $<\omega, k>^{-1}$.

Eliasson $[E_1]$, extending the method of Siegel [Si], has show by direct estimation that the series $(I.21)_\alpha$ converges. The idea is to isolate the mechanism responsible for anomolously large contributions to the tree expansion and then, introduce more refined compensating counter terms by another renormalization.

II. Ground State Energy Density

Let $x_1, \cdots, x_N$ be the positions of N electrons, each of mass m, moving in the periodic box $\mathbf{R}^d/L\mathbf{Z}^d = [-L/2, L/2]^d$, $d \geq 1$, and interacting through a real two body potential $V(x) = V(|x|)$ in $L^1 \cap L^\infty(\mathbf{R}^d)$. The Hamiltonian of our system (ignoring spin)

$$H_{N,L,\varepsilon} = \sum_{i=1}^{N} -\frac{1}{2m}\Delta_{x_i} + \frac{\varepsilon}{2}\sum_{i,j} V(x_i - x_j) \tag{II.1}$$

acts on the Hilbert space

$$F_{N,L} = \Big\{\psi \in L^2\left((\mathbf{R}^d/L\mathbf{Z}^d)^N\right) \;\Big|\; \psi(x_{\pi(1)}, \cdots, x_{\pi(N)}) = (-1)^{sgn(\pi)}\psi(x_1, \cdots, x_N) \quad \text{for all } \pi \in S_N\Big\}. \tag{II.2}$$

We want to study the thermodynamic limit

$$\mathcal{H}(\varepsilon, \rho) = \lim_{\substack{N,L\to\infty \\ \frac{N}{L^d}=\rho}} H_{N,L,\varepsilon} \tag{II.3}$$

at particle density ρ.

The Hamiltonian $\mathcal{H}(\varepsilon, \rho)$ is an unwieldy object. It is much more convenient to work with a completely equivalent description of the thermodynamic limit in terms of the Schwinger functions of an associated quantum field. However, to avoid the formalism of quantum field theory as far as possible in the present discussion, we shall restrict our attention to a simple physical quantity associated with $\mathcal{H}$ that already exhibits all of the difficulties inherent in (II.3). Namely, the ground state energy density.

The ground state energy per unit volume of $H_{N,L,\varepsilon}$ is

$$\mathcal{E}_{N,L}(\varepsilon) = \frac{\text{inf spec}\,(H_{N,L,\varepsilon})}{L^d} \quad . \tag{II.4a}$$

Expanding,

$$\mathcal{E}_{N,L}(\varepsilon) = \sum_{j\geq 0} \mathcal{E}_{j,N,L} \left(\frac{(-1)^j}{2^j j!}\right)\varepsilon^j \quad . \tag{II.4b}$$

The formal thermodynamic limit

$$
\begin{aligned}
\mathcal{E}(\varepsilon,\rho) &= \lim_{\substack{N,L\to\infty \\ \frac{N}{L^d}=\rho}} \mathcal{E}_{N,L}(\varepsilon) \\
&\sim \sum_{j\geq 0} \left(\lim_{\substack{N,L\to\infty \\ \frac{N}{L^d}=\rho}} \mathcal{E}_{j,N,L} \right) \left(\frac{(-1)^j}{2^j j!} \right) \varepsilon^j \\
&\sim \sum_{j\geq 0} \mathcal{E}_j(\rho) \left(\frac{(-1)^j}{2^j j!} \right) \varepsilon^j
\end{aligned}
\tag{II.5}
$$

is called the ground state energy density at particle density ρ. The rest of this section is devoted to the formal power series (II.5).

There is a tree expansion for $\mathcal{E}(\varepsilon,\rho)$. (See [FT1], Section 6 and [FT2], 218-223.) Indeed, it motivated our presentation of Lemma I.1. Here, unfortunately, the recipe defining the value of a tree contributing to $\mathcal{E}_j(\rho)$ is much more complicated. The value of a tree is itself expressed as a further sum over the values of vacuum graphs consistent with the tree. (See [FT2],218-223.) To compress the discussion, we shall suppress the trees and procede to write $\mathcal{E}_j(\rho)$ as the sum over the values of all vacuum graphs.

We now introduce the notion of vacuum graph to prepare for an algorithm that, roughly speaking, expresses $\mathcal{E}_j(\rho)$ as sums of integrals of products of

$$
\hat{V}(\mathbf{p}) = \int_{\mathbf{R}^d} d^d\mathbf{x}\ e^{i\mathbf{p}\cdot\mathbf{x}} V(\mathbf{x})\ , \quad \mathbf{p}\in\mathbf{R}^d \tag{II.6}
$$

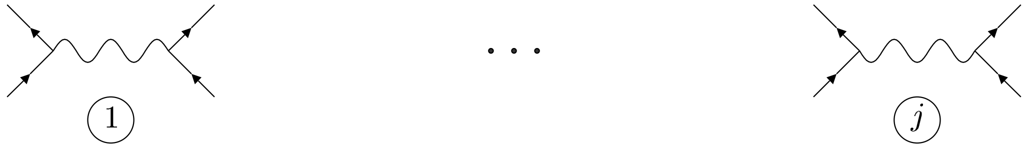
and free propagators

$$
\frac{1}{ip_0 - e(\mathbf{p})} \tag{II.7a}
$$

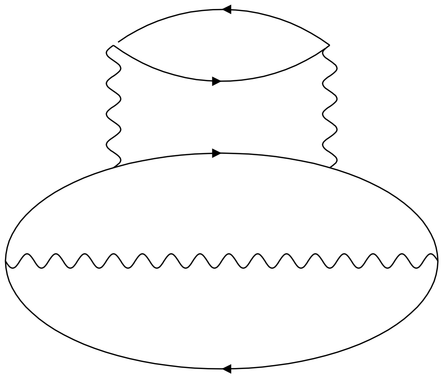
where

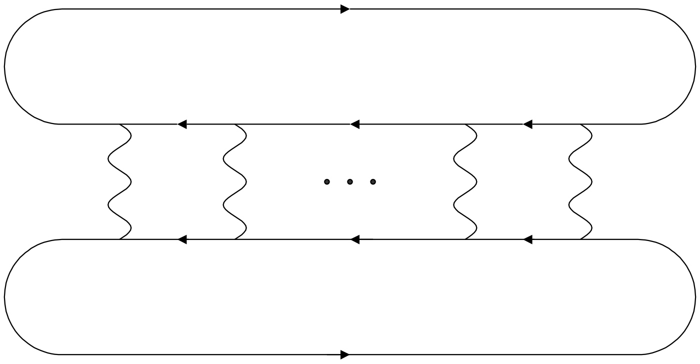
$$
e(\mathbf{p}) = \frac{|\mathbf{p}|^2}{2m} - \mu\ , \quad \mu = \frac{2\pi}{m}\left(\frac{d}{4}\Gamma(\frac{d}{2})\rho\right)^{\frac{2}{d}} \tag{II.7b}
$$

By definition, a vacuum graph of order j is constructed from j labelled vertices



by joining each of the $2j$ outgoing legs to one of the $2j$ incoming legs. There are $(2j)!$ vacuum graphs. For example,





The directed and wavy bonds are called particle and interaction lines respectively. Joining a pair of legs produces one particle line.

The lemma below takes the place of Lemma I.1 for the ground state energy density.

Lemma II.1

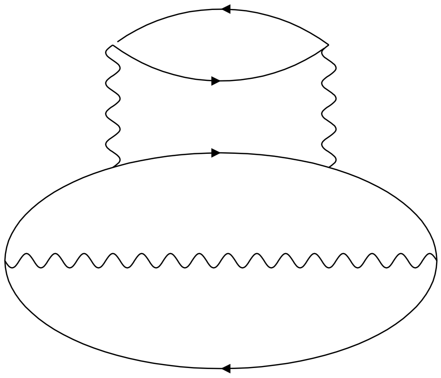
For each $j \geq 0$,

$$\mathcal{E}_j(\rho) = \sum_G val(G)$$

where the sum is over all connected vacuum graphs of order j.

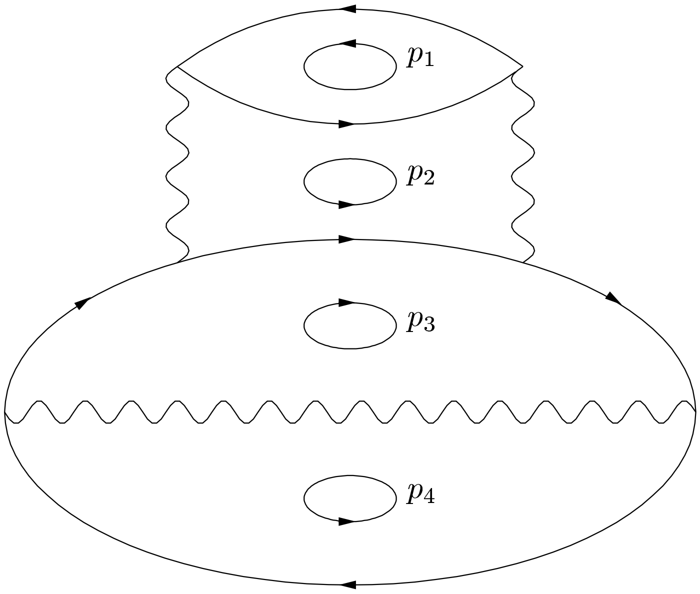
To complete the statement of Lemma II.1 the value of a vacuum graph $val(G)$ must be defined. The rules, comprising the definition, are somewhat involved and therefore best illustrated by means of an example.

We give the recipe for computing the value of the third order connected vacuum graph



$$(II.8)$$

The value of any other vacuum graph is computed in the same way. First choose a basis for the homology ("momentum loops")



$$(II.9)$$

and assign a $(d+1)$ - momentum $p = (p_0, \mathbf{p}) \in \mathbf{R}^{d+1}$ to each cycle. In (II.9) there are four independent cycles.

One imagines that momentum p flows through the particle and interaction lines forming the cycle to which it has been assigned and, that it flows in the direction induced by the orientation of the cycle. For example, in (II.9), p_2 flows up through the right vertical interaction line, from right to left through the second horizontal particle line (counting from the top), down the left vertical interaction line, and finally from left to right in the lower particle line joining the interaction lines. Note that it flows against the intrinsic direction of the upper particle line and with that of the lower one. Momentum p_1 also flows through the second horizontal particle line. In contrast to p_2 , the induced direction matches the intrinsic one.

The value of (II.8) is

$$
\begin{gathered}
(-1)^4 \int \prod_{\ell=1}^{4} \frac{d^{d+1}p_\ell}{(2\pi)^{d+1}} \frac{1}{i(p_1)_0 - e(\mathbf{p}_1)} \\
\hat{V}(\mathbf{p}_2) \frac{1}{i(p_1 - p_2)_0 - e(\mathbf{p}_1 - \mathbf{p}_2)} \hat{V}(\mathbf{p}_2) \\
\frac{1}{i(p_3)_0 - e(\mathbf{p}_3)} \frac{1}{i(p_2 + p_3)_0 - e(\mathbf{p}_2 + \mathbf{p}_3)} \frac{1}{i(p_3)_0 - e(\mathbf{p}_3)} \\
\hat{V}(\mathbf{p}_3 - \mathbf{p}_4) \\
\frac{1}{i(p_4)_0 - e(\mathbf{p}_4)}
\end{gathered} \tag{II.10}
$$

where the integrand is the product of the five rows.

Expression (II.10) is generated by, first observing that momentum p_1 flows in the same direction as the topmost directed particle line. According to the general recipe, the propagator $(i(p_1)_0 - e(\mathbf{p}_1))^{-1}$ is placed in the product forming the integrand. Next, momentum p_2 flows through both of the vertical interaction lines contributing the factors of $\hat{V}$ $(\mathbf{p}_2)$ in the second row of (II.10). The momenta p_1 and p_2 flow in opposite directions through the other directed particle line forming the particle loop, with p_1 in the same direction. Accordingly, the free propagator $(i(p_1 - p_2)_0 - e(\mathbf{p}_1 - \mathbf{p}_2))^{-1}$ is placed in the integrand. The propagator $(i(p_3)_0 - e(\mathbf{p}_3))^{-1}$ appears twice because the momentum p_3 flows, with the same orientation, through both particle lines on the shoulders. And so on. In other words, each particle line contributes a propagator, evaluated at an oriented sum of all momenta flowing through the line, to the integrand. The orientation of each momentum in the sum is determined by comparing the direction of the cycle to which it has been assigned with the intrinsic direction of the line. Plus, if they are the same, and minus, when opposite. Similarly, the interaction lines contribute factors of $\hat{V}$ evaluated at an oriented sum of d-momenta. Since $\hat{V}(\mathbf{p})$ is even, only the relative orientation matters. It is not hard to show that the value of a vacuum graph is independent of the choice of homology basis. Changing homology basis produces a linear change of variables inside the integral with Jacobian 1.

The derivation of Lemma II.1 is too lengthy to include, or even sketch, here. For an extended discussion see, for example, [FW, Chapter 3].

Write

$$val(II.8) = \int \frac{d^{d+1}p_3}{(2\pi)^{d+1}} \; T(p_3) \left(\frac{1}{i(p_3)_0 - e(\mathbf{p}_3)} \right)^2 U(p_3) \tag{II.11}$$

where

$$T(q) = \int \frac{d^{d+1}p_1}{(2\pi)^{d+1}} \frac{d^{d+1}p_2}{(2\pi)^{d+1}} \frac{1}{i(p_1)_0 - e(\mathbf{p}_1)} \hat{V}(\mathbf{p}_2) \frac{1}{i(p_1 - p_2)_0 - e(\mathbf{p}_1 - \mathbf{p}_2)} \hat{V}(\mathbf{p}_2) \frac{1}{i(p_2 + q)_0 - e(\mathbf{p}_2 + \mathbf{q})} \tag{II.12a}$$

and

$$U(q) = \int \frac{d^{d+1}p_4}{(2\pi)^{d+1}} \; \hat{V}(\mathbf{q} - \mathbf{p}_4) \; \frac{1}{i(p_4)_0 - e(\mathbf{p}_4)} \quad . \tag{II.12b}$$

The free propagator $(ip_0 - e(\mathbf{p}))^{-1}$ has a linear singularity

$$\left| \frac{1}{ip_0 - e(\mathbf{p})} \right| \approx \frac{1}{|p_0| + (\text{const}\,)\left||\mathbf{p}| - \sqrt{2m\mu}\right|} \tag{II.13}$$

on the Fermi surface

$$\left\{ p_0 = 0 \; , \quad |\mathbf{p}| = \sqrt{2m\mu} \right\} . \tag{II.14}$$

If $U(p_3)T(p_3)$ does not vanish on (II.14), then

$$U(p_3)T(p_3) \left(\frac{1}{i(p_3)_0 - e(\mathbf{p}_3)} \right)^2$$

has a nonintegrable singularity on the Fermi surface and $val(II.8) = \infty$.

Observe that $T(q_0, R\mathbf{q}) = T(q_0, \mathbf{q})$ for all R in $O(d)$. In particular, $T(q_0, \mathbf{q})$ is constant on the Fermi surface. Using residues to perform the $(p_1)_0$ and $(p_2)_0$ integrals

$$
\begin{aligned}
T(q) = & \int \frac{d^{d+1}p_2}{(2\pi)^{d+1}} \; \frac{\left|\hat{V}(\mathbf{p}_2)\right|^2}{i(p_2+q)_0 - e(\mathbf{p}_2+\mathbf{q})} \\
& \qquad \int_{e(\mathbf{p}_1)e(\mathbf{p}_1-\mathbf{p}_2)<0} \frac{d^d\mathbf{p}_1}{(2\pi)^d} \frac{sgn\,(e(\mathbf{p}_1))}{i(p_2)_0 - (e(\mathbf{p}_1) - e(\mathbf{p}_1-\mathbf{p}_2))} \\
= & \int_{\substack{e(\mathbf{p}_1)e(\mathbf{p}_1-\mathbf{p}_2)<0 \\ (e(\mathbf{p}_1)-e(\mathbf{p}_1-\mathbf{p}_2))e(\mathbf{p}_2+\mathbf{q})<0}} \frac{d^d\mathbf{p}_1}{(2\pi)^d}\frac{d^d\mathbf{p}_2}{(2\pi)^d} \; \frac{\left|\hat{V}(\mathbf{p}_2)\right|^2}{-iq_0 - (e(\mathbf{p}_1) - e(\mathbf{p}_1-\mathbf{p}_2) - e(\mathbf{p}_2+\mathbf{q}))}
\end{aligned}
\tag{II.15}
$$

One can show that $T(q)$ is a continuous function on $\mathbf{R}^{d+1}$. By (II.15)

$$
T(q_0=0, |\mathbf{q}| = \sqrt{2m\mu}) =
$$

$$
-\int_{\substack{e(\mathbf{p}_1)e(\mathbf{p}_1-\mathbf{p}_2)<0 \\ (e(\mathbf{p}_1)-e(\mathbf{p}_1-\mathbf{p}_2))e(\mathbf{p}_2+\sqrt{2m\mu}\;\mathbf{q}/|\mathbf{q}|)<0}} \frac{d^d\mathbf{p}_1}{(2\pi)^d}\frac{d^d\mathbf{p}_2}{(2\pi)^d} \; \frac{\left|2\pi\hat{V}(\mathbf{p}_2)\right|^2}{\left(e(\mathbf{p}_1) - e(\mathbf{p}_1-\mathbf{p}_2) - e(\mathbf{p}_2+\sqrt{2m\mu}\;\mathbf{q}/|\mathbf{q}|)\right)}
$$

The constraints

$$
e(\mathbf{p}_1)e(\mathbf{p}_1-\mathbf{p}_2) < 0 \;,\; (e(\mathbf{p}_1) - e(\mathbf{p}_1-\mathbf{p}_2))e(\mathbf{p}_2 + \sqrt{2m\mu}\;\mathbf{q}/|\mathbf{q}|)) < 0
$$

force the denominator to be of the same sign as $e(\mathbf{p}_1)$. Consequently, the integrand is positive when $\mathbf{p}_1$ is outside the Fermi surface and negative when it is inside. Therefore,

$$
T(q_0=0, |\mathbf{q}| = \sqrt{2m\mu}) \neq 0 \tag{II.16a}
$$

for a generic interaction V . Also, by definition,

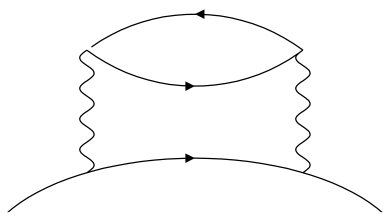
$$
\begin{aligned}
U(q) &= \lim_{\epsilon \downarrow 0} \int \frac{d^{d+1}p_4}{(2\pi)^{d+1}} \; \hat{V}(\mathbf{q}-\mathbf{p}_4) \; \frac{e^{i(p_4)_0\epsilon}}{i(p_4)_0 - e(\mathbf{p}_4)} \\
&= \int_{e(\mathbf{p}_4)<0} \frac{d^d\mathbf{p}_4}{(2\pi)^d} \; \hat{V}(\mathbf{q}-\mathbf{p}_4)
\end{aligned}
$$

so that

$$
U(q_0=0, |\mathbf{q}| = \sqrt{2m\mu}) \neq 0 \tag{II.16b}
$$

for a generic interaction V . Combining (II.16) with the discussion above, we conclude that $val(II.8) = \infty$ for generic V .

The mechanism of the last two paragraphs is quite general. Observe that $T(p_3)$ is the value of the two-legged subgraph

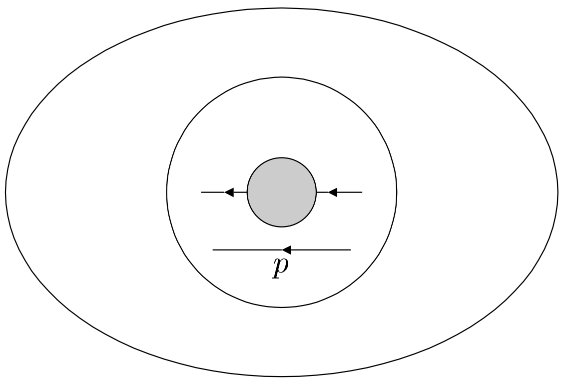


$$(II.17)$$

of (II.8).

To be precise, a $2n$-legged graph of order j is constructed as before from j labelled vertices by joining all but n of the $2j$ outgoing legs to one of the $2j$ incoming legs. Of course, there are n incoming legs left over. The value of a $2n$-legged graph is computed by choosing a basis for the homology, a distinguished external leg and $2n-1$ additional paths, each joining an external leg to the distinguished one. It is a function (distribution) of the momenta flowing in the $2n-1$ paths. (See, [FT1, p.165]) In (II.17), momentum p_3 flows in from the left along the bottom most particle line and out to the right.

Now, let



$$(II.18)$$

be any vacuum graph with momentum p flowing along a cycle through a two-legged subgraph. Then

$$val(II.18) = \int \cdots \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \left(\frac{1}{ip_0 - e(\mathbf{p})}\right)^2 val(\text{subgraph})(p)\ f(\cdots\ p).$$

We see that $val(II.18)$ is potentially divergent because

$$\left(\frac{1}{ip_0 - e(\mathbf{p})}\right)^2 \; val(\text{subgraph})(p) \; \int \cdots \; f(\cdots \; p).$$

may have a nonintegrable singularity on the Fermi surface.

We know that any vacuum graph with a two-legged subgraph is potentially divergent. Are there are other, perhaps more subtle, reasons for the value of a graph to diverge?

<u>Theorem II.1</u> ([FT1], [FT2])

The value of any vacuum graph, without two-legged subgraphs, is finite.

It is implicit in Theorem II.1 that each order of perturbation theory is "ultraviolet finite". Concretely, let

$$h(x) = \begin{cases} 0, & x \le 1 \\ 1, & x \ge 2 \end{cases}$$

be a smooth function interpolating between 0 and 1, and

$$\frac{h(p_0^2 + e(\mathbf{p})^2)}{ip_0 - e(\mathbf{p})} \qquad (II.19).$$

the ultraviolet (large momentum) part of the free propagator (II.7). In particular, the cutoff $h(p_0^2 + e(\mathbf{p})^2)$ excises a shell around the Fermi surface eliminating the singularity (II.14). In the course of proving Theorem II.1, one shows that the ultraviolet value of any vacuum graph, in which (II.7a) is replaced by (II.19), is convergent, but not necessarily absolutely convergent. We may say that each graph is ultraviolet finite, but potentially infrared divergent when and only when it contains a two-legged subgraph. Later, we will formulate a quantitative version of Theorem II.1.

The "infrared" divergent portions of (II.18)

$$\left(\frac{1}{ip_0 - e(\mathbf{p})}\right)^2 \; val(\text{subgraph})(p)$$

can be removed by introducing a counterterm

$$val(\text{subgraph})(p\prime) = val(\text{subgraph})(p_0 = 0, \frac{\mathbf{p}}{|\mathbf{p}|}\sqrt{2m\mu}) \qquad (II.20)$$

for each <u>two-legged</u> subgraph, where

$$p\prime = \ (p_0 = 0, \frac{\mathbf{p}}{|\mathbf{p}|}\sqrt{2m\mu}) \tag{II.21}$$

is the projection onto the Fermi surface. It is a number, because

$$val(\text{subgraph})(p_0, R\mathbf{p})) = val(\text{subgraph})(p_0, \mathbf{p}))$$

for all R in $O(d)$. Subtracting, the integral

$$\int \cdots \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \left(\frac{1}{ip_0 - e(\mathbf{p})}\right)^2 (val(\text{subgraph})(p) - val(\text{subgraph})(p\prime)) \ f(\cdots \ p).$$

becomes finite because

$$\left(\frac{1}{ip_0 - e(\mathbf{p})}\right)^2 (val(\text{subgraph})(p) - val(\text{subgraph})(p\prime))$$

now has an integrable singularity on the Fermi surface. To apply Taylor's theorm, it is necessary to check that $val(\text{subgraph})(p)$ is sufficiently regular.

In the last paragraph we implicitly assumed that all the two-legged subgraphs had well-defined values. The idea, loosely speaking, is to start with first order graphs and proceed inductively to define the renormalized value, $val_{ren}(G)$, of each j-th order graph. The first nontrivial example is the third order graph (II.8). We subtract the counterterm (II.16) and define its renormalized value by

$$val_{ren}(II.8) =$$

$$\int \frac{d^{d+1}p_3}{(2\pi)^{d+1}} \frac{d^{d+1}p_4}{(2\pi)^{d+1}} \ (T(p_3) - T(p\prime_3)) \left(\frac{1}{i(p_3)_0 - e(\mathbf{p}_3)}\right)^2 \hat{V}(\mathbf{p}_3 - \mathbf{p}_4) \ \frac{1}{i(p_4)_0 - e(\mathbf{p}_4)} \tag{II.22}$$

At order j , a two-legged subgraph is of order at most $j-1$. By induction, it has a well-defined renormalized value. Therefore, the associated counterterm is also well-defined. Subtracting, as above, we can construct the finite renormalized value of each j-th order graph. Some work is required to implement this scheme. In [FT1] and [FT2] it is recast as the iteration of a renormalization group map. This is a convenient setting in which to perform the estimates.

By definition, the renormalized ground state energy density is the formal power series

$$\begin{aligned}\mathcal{E}_{ren}(\varepsilon,\rho) &\sim \sum_{j\geq 0} \mathcal{E}_{ren\ j}(\rho) \ \left(\frac{(-1)^j}{2^j j!}\right)\varepsilon^j \\ &= \sum_{j\geq 0}\left(\sum_G \ val_{ren}(G,\mu)\right)\left(\frac{(-1)^j}{2^j j!}\right)\varepsilon^j\end{aligned} \tag{II.23}$$

where the sum $\sum_G$ is over all connected vacuum graphs. By construction, $\mathcal{E}_{ren\ j}(\rho)$ is finite at every order and a real analytic function of $\mu > 0$. Clearly, (II.23) plays the role of $(I.21)_\alpha$. As in Section I, this remedy for infrared divergence might be unacceptable, because $\mathcal{E}_{ren}(\varepsilon,\rho)$, while well-defined, is not, on the face of it, the "true" ground state energy density. However, it is justified, because it can be implemented by renormalizing the density ρ (or equivalently, μ). That is, choosing μ as a function (formal power series) of ε in a specific way.

Theorem II.2 ([FT1], [FT2])

Fix

$$\mu_0 = \frac{2\pi}{m}\left(\frac{d}{4}\Gamma(\frac{d}{2})\rho_0\right)^{\frac{2}{d}}.$$

Then, there is a unique, well-defined formal power series

$$\mu(\varepsilon) = \sum_{j\geq 0} \ \mu_j \ \varepsilon^j$$

such that

$$\begin{aligned}\mathcal{E}(\varepsilon,\rho(\varepsilon)) &\sim \sum_{j\geq 0} \mathcal{E}_{ren\ j}(\rho_0) \ \left(\frac{(-1)^j}{2^j j!}\right)\varepsilon^j \\ &= \sum_{j\geq 0}\left(\sum_G \ val_{ren}(G,\mu_0)\right)\left(\frac{(-1)^j}{2^j j!}\right)\varepsilon^j\end{aligned}$$

is a well-defined formal power series expansion for the ground state energy density at density $\rho(\varepsilon)$.

We remark that the proof of Theorem II.2 does not require an argument analogous to (I.26,27,28,29). In Section I, the (infinite dimensional) group of formal symplectic maps was used to show that when the d action parameters were exhausted by the first d counterterms, the the second d counterterms automatically vanished Here, both the counterterms at order j and μ_j are scalars, so there is no mismatch. However, in the renormalization of a gauge field there again appear to be too many counterterms. In that context the (infinite dimensional) group of gauge transformations generates Ward identities that force the "extra" counterterms to vanish.

Our discussion of the ground state energy density and the accompaning vacuum graphs led us naturally to two-legged graphs and their values. For this reason we consider the analogous formal power series

$$G_1(q_1) \sim \sum_{j\geq 0} \left(\sum_G \; val(G)(q_1) \right) \; \left(\frac{(-1)^j}{2^j j!} \right) \varepsilon^j$$

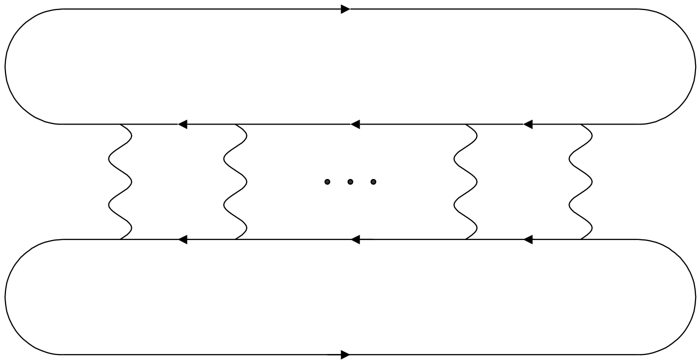
where the internal sum is over all connected two-legged graphs, and in general,

$$G_n(q_1, \cdots, q_{2n-1}) \sim \sum_{j\geq 0} \left(\sum_G \; val(G)(q_1, \cdots, q_{2n-1}) \right) \left(\frac{(-1)^j}{2^j j!} \right) \varepsilon^j \qquad (II.24)_n$$

where the internal sum is now over all 2n-legged graphs for which each connected component contains an external leg. The "functions" G_n , $n \geq 1$, are called the Schwinger functions. They characterize the thermodynamic limit. In fact, there is a standard procedure for constructing a Hilbert space and a semi-group from them. The generator of the semi-group is the Hamiltonian (II.3). The formal power series defining the Green's functions are ill-defined for exactly the same reason as (II.5) and may be renormalized in the same way.

To this point, though considerably more involved technically, the perturbative analysis of the ground state energy density (and, by the preceding remark, of the thermodynamic limit (II.3) itself) is a close parallel to the contruction of formal quasi-periodic orbits. There is a further similarity. The values of some graphs contributing to $\mathcal{E}_{ren\ j}$ are just "too big".

Consider the j-th order "right-way ladder"



$$(II.25j)$$

By direct calculation, one can show (See, [FT1, p.197]) that

$$val(II.25j) \ \approx \ (\text{const}\,)^j \ j!\ . \tag{II.26}$$

(Note that $val(II.25j) = val_{ren}(II.25j)$ since (II.25j) has no two-legged subgraphs.) Recall that we formed graphs from <u>labelled</u> vertices. So, there are many graphs that look just like (II.25j) when the labels are erased. Counting carefully, one finds exactly $2^{j-1}\ j!$ of them. They all have the same value. Extracting these graphs, and forming the sum

$$\sum_{j\ge 0} 2^{j-1}\ j!\ val(II.25j)\ \left(\frac{(-1)^j}{2^j j!}\right)\varepsilon^j \ \approx\ \sum_{j\ge 0}\frac{1}{2}j!\ (\text{const}\,)^j(-\varepsilon)^j \tag{II.27}$$

we see that there is a piece of $\mathcal{E}(\varepsilon,\rho(\varepsilon))$ that diverges for all ε different from 0 . The subseries (II.27) should be compared with (I.48).

The divergence of (II.27) warns us that the series

$$\mathcal{E}(\varepsilon,\rho(\varepsilon)) \sim \sum_{j\ge 0}\ \mathcal{E}_{ren\ j}(\rho_0)\ \left(\frac{(-1)^j}{2^j j!}\right)\varepsilon^j$$

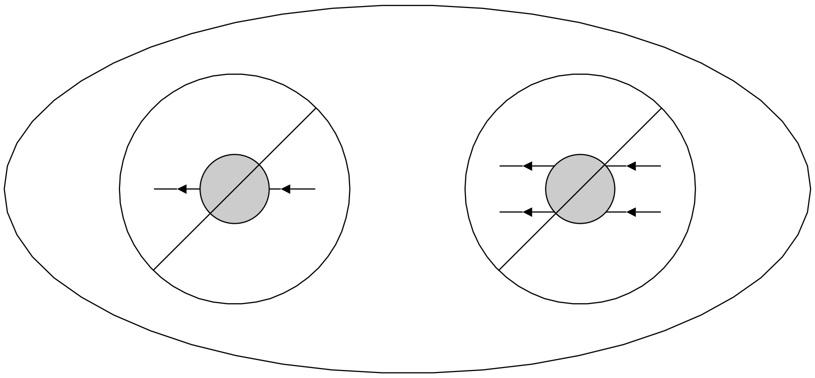
of Theorem II.2 may itself diverge. In fact, when the Hamiltonian (II.1) is supplemented by the interaction between electrons and the vibrations of a background lattice of ions (that is, phonons) the series cannot converge. (See, [FT2]) Then, the free ground state is unstable in the thermodynamic limit, and one cannot, as we have done, naively expand about the free Hamiltonian

$$\mathcal{H}_0(\varepsilon,\rho) = \lim_{\substack{N,L\to\infty\\ \frac{N}{L^d}=\rho}} \left\{\sum_{i=1}^{N}\ -\frac{1}{2m}\Delta_{x_i}\ \text{ acting on }\ F_{N,L}\right\}\ .$$

The instability is caused by a phase transition to a superconducting ground state.

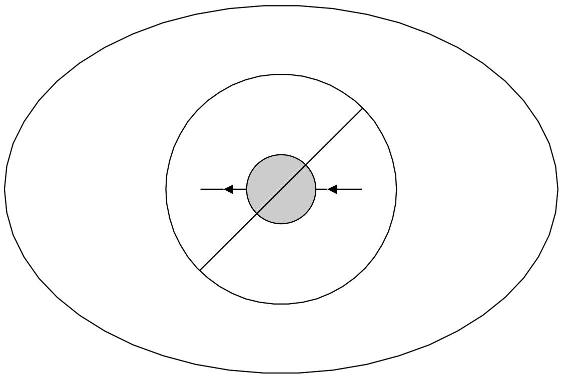
In Section 1, the series of Theorem I.1 converge in spite of anomalously large contributions to the tree expansion (See the remark following (I.48)). There are cancellations between the values of different trees. Here, by contrast, the series of Theorem II.1 is forced to diverge by the large values of certain graphs.

To see what is involved consider the following quantitative version of Theorems II.1 and II.2. Let



$$(II.28)$$

be a connected vacuum graph of order j that *contains no nontrivial two or four-legged subgraphs* and similarly, let

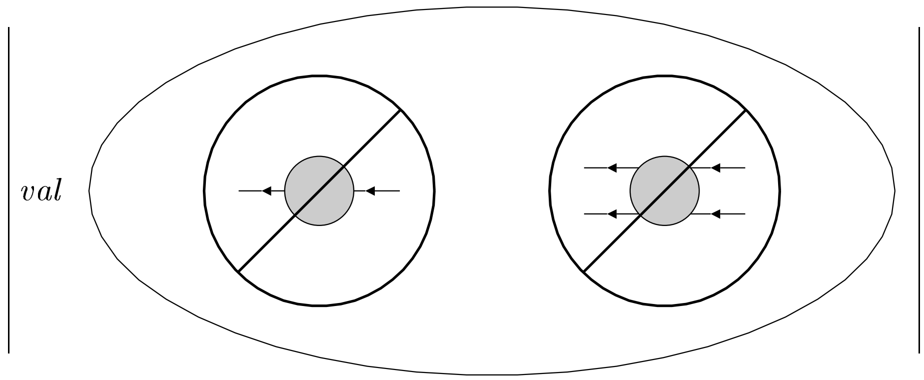


$$(II.29)$$

be a graph *without two-legged subgraphs.* Notice that the ladder (II.25j) contains many four-legged subgraphs.

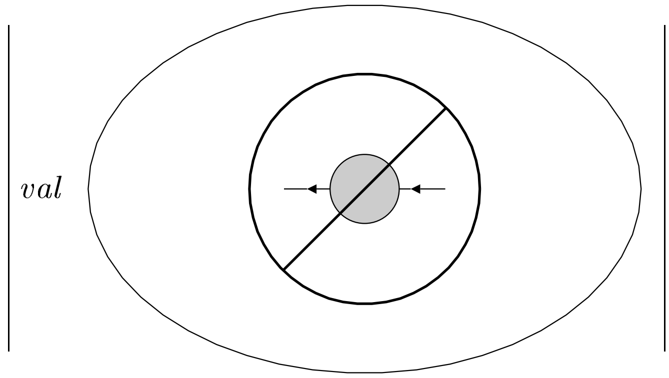
Theorem II.3 ([FT1], [FT2])

(i)



$\leq$ (const)^j

(ii)



$\leq$ (const)$^j N_F!$

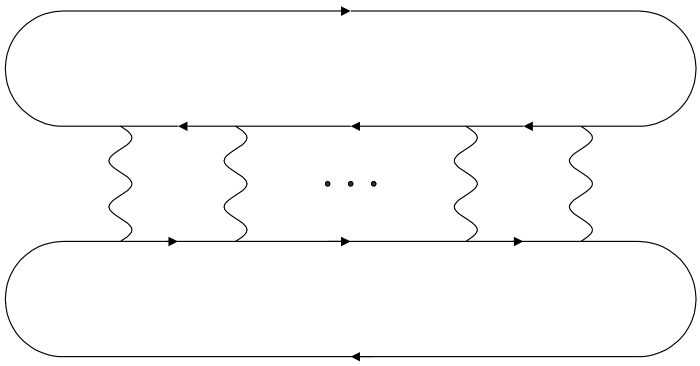
where N_F is the maximal number of nonoverlapping, nontrivial four-legged subgraphs.

(iii) Let G be any connected vacuum graph. Then

$$val_{ren}(G) \ < \ (\text{const}\,)^j \ N_F!$$

The constants in (i),(ii) and (iii) are <u>independent</u> of the graph. Similar statements for 2n-legged graphs hold, when the size $|\cdots|$ is measured by an appropriate norm.

Theorem II.3 is important. We now know that the <u>only</u> way to produce an effect like (II.27) is to accumulate four-legged subgraphs. However, as the strict inequality of (iii) indicates,the appearance of many four-legged subgraphs does not guarantee that the value is large. For example, the j-th order "wrong-way ladder"



$$(II.30j)$$

is only $O\left((\text{const}\,)^j\right)$.

We also learn from Theorem II.3, that the value of a graph without two and four-legged subgraphs (or, the renormalized value of a graph without four-legged subgraphs) is not only finite, but the "right size". The "right size", in the sense that, it contributes a $(\text{const}\,)^j$ at order ε^j in the expansion of $\mathcal{E}(\varepsilon, \rho(\varepsilon))$ and is therefore, "morally irrelevant" when ε is small.

<u>Suppose</u> the size of every graph were $O(\text{const}^{\,j})$, then the naive estimate

$$\sum_{j\geq 0} \frac{(2j)!}{2^j\ j!}\,(\text{const}\,)^j\varepsilon^j \ \sim \sum_{j\geq 0}\ j!\ (\text{const}\,)^j\varepsilon^j$$

diverges because there are $O((2j)!) \approx O(\text{const}^j\ (j!)^2)$ graphs. The large number of graphs is a new feature. We saw, in Section 1, that, assuming Theorem I.2 for all trees and momenta assignments, the tree expansion converges because there are at most 4^j rooted planar trees. This phenomenon is typical of systems with infinitely many degrees of freedom. To prove convergence, cancellations between graphs must also be exploited.

Large graphs are handled in [FT2] by first carefully identifying the portions of four-legged subgraphs that are responsible for their anomalous size. See [FT2],(I.29) and (I.99). Next, we decompose the Hamiltonian ([FT2], (I.64))into new "free" and "interacting" parts. The new "free propagator" is

$$\frac{1}{ip_0\mathbb{1}\ -\ e(\mathbf{p})\sigma^3\ -\ \Delta\sigma^1} \tag{II.31}$$

where Δ is the solution of a BCS-like gap equation ([FT2, (I.75)) and σ^1, σ^3 are Pauli matrices. Finally, the above portions of four-legged subgraphs are resummed by means of a renormalization group flow that converges to a nontrivial (superconducting) fixed point near (II.31) ([FT2], Theorem I.3). This construction is too elaborate to discuss here. After resummation, the value of each graph is $O(\mathrm{const}^{\,j})$.

Bibliography

[A1] Arnold, V.I., Mathematical Methods of Classical Mechanics, Graduate Texts in Mathematics 60 Springer-Verlag (1980)

[A2] Arnold, V.I., Small denominators II, Proof of a theorem of A.N. Kolomogorov on the preservation of conditionally-periodic motions under a small perturbation of the Hamiltonian, Russian Math. Surveys 18:5 (1963)

[E1] Eliaisson, H Absolutely Convergent Series Epansions for Quasi Periodic Motions Stockholm preprint (1988)

[E1] Eliaisson, H Generalization of an estimate of Small Divisors by Siegel in "Analysis, Et Cetera" ed Rabinowitz,Zehnder Academic Press (1990)

[FT1] Feldman, J., Trubowitz, E., Perturbation Theory for Many Fermion Systems, Helvetica Physica Acta Vol. 63 (1990)

[FT2] Feldman, J., Trubowitz, E., The Flow of an Electron-Phonon System to the Superconducting State, Helvetica Physica Acta Vol. 64 (1991)

[FW] Fetter, A.L., Walecka, J.D., Quantum Theory for of Many-Particle Systems, McGraw-Hill, (1971)

[GJ] Goulden, I.P., Jackson, D.M., Combinatiorial Enumeration John Wiley (1983)

[K] Kolmogorov, A.N., On the preservation of conditionally-periodic motions for small change in Hamilton's function, Dokl. Akad. Nauk SSSR 98:4 (1954)

[P] Poincaré, H., Les Méthodes Nouvelles de la Méchanique Cleleste, Vol. 1 Dover, (1957)

[Sc] Schmidt, Wolfgang, M. Diophantine Approximation Lecture Notes in Mathematics 785 Springer-Verlag (1980)

[Si] Siegel, C.L. Iteration of Analytic Functions, Annals of Mathematics 43 (1942)