Evaluation of Fermion Loops by Higher Residues

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* Research supported in part by the Natural Sciences and Engineering Research Council of Canada, the Schweizerischer Nationalfonds zur Förderung der wissenschaftlichen Forschung and the Forschungsinstitut für Mathematik, ETH Zürich
† Research supported in part by the IHES and SFB288 Differentialgeometrie und Quantenphysik (Berlin)
§I Introduction

Let

\[ e(k) = \frac{1}{\pi} |k|^2 - \mu, \quad k \in \mathbb{R}^2 \]

be the dispersion relation for a two dimensional electron gas with chemical potential \( \mu > 0 \).

By definition, the amplitude of the \((n+1)\) Fermion loop with external momenta \( q_i = (q_{i0}, q_i) \in \mathbb{R} \times \mathbb{R}^2, i = 1, 2, \ldots, n + 1 \), is

\[
I(q_1, \ldots, q_{n+1}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} dk_1 dk_2 \int_{-\infty}^{+\infty} dk_0 \frac{dk_0}{\prod_{i=1}^{n+1} \left( \alpha(k - q_i) - e(k - q_i) \right)}
\]

Recall that the amplitude of an arbitrary diagram contributing to the formal power series expansion of the corresponding many fermion Green’s functions in powers of the coupling constant is obtained by integrating products of Fermion loops and interactions.

It is not hard to evaluate the 2-loop (the “polarization bubble”) \( I(q_1, q_2) \) when \((q_1 - q_2)_0 \neq 0\) and \( q_1 - q_2 \neq 0 \). One finds (see, for example, [FKST], Proposition II.1)

\[
I(q_1, q_2) = -\frac{m}{2\pi} + \frac{mk_F}{2\pi|q_1 - q_2|} \Re \left( \alpha - \frac{1}{\alpha} \right)
\]

where \( \alpha \) is the root outside the unit circle of

\[
z^2 - \frac{1}{k_F} \left( |q_1 - q_2| - 2m \frac{(q_1 - q_2)_0}{|q_1 - q_2|} \right) z + 1 = 0
\]

and \( k_F = \sqrt{2m\mu} \). The zero frequency limit has a particularly interesting behaviour. Namely,

\[
I\left((0, q_1), (0, q_2)\right) = \begin{cases} 
\frac{-m}{2\pi}, & 0 \leq |q_1 - q_2| \leq 2k_F \\
\frac{-m}{2\pi} \left( 1 - \frac{1}{|q_1 - q_2|} \sqrt{|q_1 - q_2|^2 - 4k_F^2} \right), & |q_1 - q_2| \geq 2k_F
\end{cases}
\]

In particular, the value of the 2-loop is \(-\frac{m}{2\pi}\) whenever the disks of radius \( k_F \) centered at \( q_1 \) and \( q_2 \) overlap.

In this paper, we evaluate the \((n+1)\) Fermion loop explicitly for all \( n \geq 2 \). Our result (see, Theorem III.1 and the Remarks after it) is

\[
I(q_1, \cdots, q_{n+1}) = \frac{m^n}{2\pi^2} \sum_{i,j=1 \atop i \neq j}^{n+1} \int w_{ij} \varphi_{ij}(z) \, dz
\]
where $\varphi_{ij}$ is an explicitly computable rational function of one complex variable and $w_{ij}$ is an explicitly given curve in $\mathbb{C}$. For $n = 2$, that is the 3-loop, we have written out the function $\varphi_{ij}$ and the path $w_{ij}$ in all detail.

Again, the zero frequency limit

$$J(p_1, \ldots, p_{n+1}) = \lim_{q_i \to (p_i, p_i)} I(q_1, \ldots, q_{n+1})$$

is interesting. The third author (see, [Si]) evaluated $J(p_1, p_2, p_3)$ numerically and observed that

$$J(p_1, p_2, p_3) = 0$$

whenever the disks of radius $k_F$ centered at $p_1$, $p_2$ and $p_3$ have a point in common. We have evaluated (Theorem III.2) $J(p_1, \ldots, p_{n+1})$. In particular (Corollary III.3 (ii)), for all $n \geq 2$,

$$J(p_1, \ldots, p_{n+1}) = 0$$

when all $n + 1$ disks with radius $k_F$ around the points $p_1, \ldots, p_{n+1}$ have at least one point in common. * Also (Corollary III.3 (iii)),

$$J(p_1, p_2, p_3) = \frac{m^2}{2 \times \text{area of the triangle with vertices } p_1, p_2, p_3}$$

when the three disks with radius $k_F$ around the points $p_1, p_2, p_3$ have no point in common but any two of the disks intersect.

To evaluate $I(q_1, \ldots, q_{n+1})$, we perform (see, the beginning of §III) the integral over $k_0$ to obtain

$$I(q_1, \ldots, q_{n+1}) = \frac{m^n}{(2\pi)^2} \sum_{i=1}^{n+1} \int_{|k-q_i|<k_F} \frac{dk_1 dk_2}{\prod_{j=1 \atop j \neq i}^{n+1} f_{ij}(k)}$$

where

$$f_{ij}(k) = (q_j - q_i) \cdot k + \frac{1}{2} (q_i^2 - q_j^2) + i m (q_{i0} - q_{j0})$$

* The Appendix contains an elementary proof of this fact for the special case that $n = 2$ and the triangle with vertices $p_1, p_2, p_3$ is acute.
Next, it is observed that each summand

$$\int_{|k - q_i| < k_F} \frac{dk_1 dk_2}{\prod_{j=1}^{n+1} f_{ij}(k)}$$

is of the generic form

$$\int_{D_\rho(q)} \frac{dx_1 \wedge dx_2}{\prod_{j=1}^{n} (a_j x_1 + a_j x_2 - b_j)}$$

where the coefficients $a_j, a_j, j = 1, \cdots, n$, are real and $b_j$ is complex. Here, $D_\rho(q)$ is the disk of radius $\rho > 0$ centered at $q$. In §III, we replace the disk by a homologous cycle in the complex projective plane $\mathbb{P}^2$ where higher residues can be applied.

It is a great pleasure to thank Andrea Cavalli and Peter Wagner for many helpful suggestions. Peter Wagner also found a different proof of Theorem III.2, using methods similar to those of [OW]. He has generously allowed us to incorporate some of his improvements in the statement of Theorem III.2 itself.
§II The Integral of a Particular Rational Function over a Disk

Let \( n \geq 2 \) and fix a real matrix

\[
A = \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22} \\
  \vdots & \vdots \\
  a_{n1} & a_{n2}
\end{pmatrix}
\]

such that no two by two subdeterminant vanishes. Fix \( b \in \mathbb{C}^n \) such that \( \Im b_j \neq 0 \), for all \( j = 1, \ldots, n \). For each \( \mathbf{q} \in \mathbb{R}^2 \) and \( \rho \geq 0 \) let

\[
D_\rho(\mathbf{q}) = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - \mathbf{q}_1)^2 + (x_2 - \mathbf{q}_2)^2 \leq \rho^2 \right\}
\]

be the closed disk of radius \( \rho \) with the standard orientation, centered at \( \mathbf{q} \). We shall compute

\[
\int_{D_\rho(\mathbf{q})} \frac{dx_1 \wedge dx_2}{\prod_{j=1}^{n} (a_{j1}x_1 + a_{j2}x_2 - b_j)}
\]

To formulate the result, let \( L_j, \ j = 1, \ldots, n \), be the line in the complex projective plane \( \mathbb{P}^2 \) given by

\[
L_j = \left\{ [z_0, z_1, z_2] \in \mathbb{P}^2 \mid a_{j1}z_1 + a_{j2}z_2 = b_jz_0 \right\}
\]

and let

\[
L_\infty = \left\{ [z_0, z_1, z_2] \in \mathbb{P}^2 \mid z_0 = 0 \right\}
\]

We assume that the intersections

\[
L_\alpha \cap L_\beta = \{d^{\alpha, \beta}\}
\]

for \( \alpha, \beta = 1, \ldots, n, \infty \) with \( \alpha \neq \beta \) are pairwise different and do not lie on the projective quadric

\[
Q_s = \left\{ [z_0, z_1, z_2] \in \mathbb{P}^2 \mid (z_1 - \mathbf{q}_1z_0)^2 + (z_2 - \mathbf{q}_2z_0)^2 = s^2z_0^2 \right\}
\]

for either \( s = 0 \) or \( s = \rho \).

If \( n \geq 3 \), let \( \eta_j, \ j = 1, \ldots, n \), be the unique rational one form on the projective line \( L_j \) that has a simple pole at the point \( d^{j,k} \) for all \( k = 1, \ldots, n \), \( k \neq j \), with residue

\[
\frac{1}{\det \begin{pmatrix} a_{j1} & a_{j2} \\ a_{k1} & a_{k2} \end{pmatrix}} \prod_{\ell \neq j, k} \left( \frac{1}{d_{\ell 0}} (a_{\ell 1}d_{1\ell}^{j,k} + a_{\ell 2}d_{2\ell}^{j,k} - b_{\ell k}d_{k\ell}^{j,k}) \right)^{-1}
\]
no other poles, and a zero of order \( n - 3 \) at \( d^{j, \infty} \). We will see below that the sum of these residues is zero and for this reason the form \( \eta_j \) does in fact exist. If \( n = 2 \), let \( \eta_j, j = 1, 2 \), be the unique rational one form on the projective line \( L_j \) that has a simple pole at the point \( d_j^k \) for \( k \neq j \), with residue

\[
\frac{1}{\det \left( \begin{array}{cc} a_{j1} & a_{j2} \\ a_{k1} & a_{k2} \end{array} \right)}
\]

a simple pole at \( d^{j, \infty} \), and no other poles.

For all \( s \in \mathbb{C} \), set

\[
Q_s^+ = \left\{ [z_0, z_1, z_2] \in Q_s \mid \exists \frac{z_2 - \eta_2}{z_1 - \eta_1} > 0 \right\}
\]

\[
Q_s^- = \left\{ [z_0, z_1, z_2] \in Q_s \mid \exists \frac{z_2 - \eta_2}{z_1 - \eta_1} < 0 \right\}
\]

We will show, see Lemma II.2, that for all \( s \geq 0 \) and all \( j = 1, \ldots, n \), the intersection \( Q_s^+ \cap L_j \) consists of exactly one point, say, \( c_j^+(s) \) and the intersection \( Q_s^- \cap L_j \) consists of exactly one point, say, \( c_j^-(s) \).

**Theorem II.1** Suppose that for each \( j = 1, \ldots, n \), the paths \( c_j^+(s), 0 \leq s \leq \rho \), do not pass through any of the double points \( L_j \cap L_k, j, k = 1, \ldots, n, j \neq k \). Then,

\[
\int_{D_{r(q)}} \frac{dx_1 \wedge dx_2}{\prod_{j=1}^n (a_{j1}x_1 + a_{j2}x_2 - b_j)} = -2\pi i \sum_{j=1}^n \int_{\{c_j^+(s) \mid 0 \leq s \leq \rho \}} \eta_j
\]

\[
= +2\pi i \sum_{j=1}^n \int_{\{c_j^-(s) \mid 0 \leq s \leq \rho \}} \eta_j
\]

**Remark.** Observe that for any path \( s(\sigma), 0 \leq \sigma \leq 1 \), in \( \mathbb{C} \) joining 0 to \( \rho \) and lying sufficiently close to the interval \( [0, \rho] \), the intersection \( Q_s^+ \cap L_j \) consists of exactly one point \( c_j^+(s(\sigma)) \). Our hypotheses imply that \( c_j^+(0) \) and \( c_j^+(\rho) \) are not double points \( L_j \cap L_k \). It is therefore possible to choose a path \( s(\sigma), 0 \leq \sigma \leq 1 \), arbitrarily close to \( [0, \rho] \) such that
$c_j^\pm(s(\sigma))$ is not a double point for all $j = 1, \cdots, n$ and $0 \leq \sigma \leq 1$. With this choice

$$
\int_{D_\rho(q)} \frac{dx_1 \wedge dx_2}{\prod_{j=1}^n (a_{j1}x_1 + a_{j2}x_2 - b_j)} = 2\pi i \sum_{j=1}^n \int \left\{ c_j^+(s(\sigma)) \mid 0 \leq \alpha \leq 1 \right\} \eta_j
$$

$$
= -2\pi i \sum_{j=1}^n \int \left\{ c_j^-(s(\sigma)) \mid 0 \leq \alpha \leq 1 \right\} \eta_j
$$

Two lemmas are required for the proof of Theorem II.1. Observe that by translation invariance, we may assume that $q = 0$.

**Lemma II.2** For all $s \geq 0$ and all $\alpha = 1, \cdots, n, \infty$, the intersection $Q_s^+ \cap L_\alpha$ consists of exactly one point, say, $c^+(s)$ and the intersection $Q_s^- \cap L_\alpha$ consists of exactly one point, say, $c^-(s)$. Furthermore, $c^+_\infty(s)$ and $c^-\infty(s)$ do not depend on $s$.

**Proof:** First, fix $1 \leq j \leq n$. We can rotate $(z_1, z_2)$ around zero to make $a_{j2} = 0$. The intersection $Q_s \cap L_j$ is determined by the equation

$$
z_1^2 + z_2^2 = s^2 \left( \frac{a_{j1}}{b_j} \right)^2 z_1^2
$$

Thus,

$$
\left( \frac{z_2}{z_1} \right)^2 = s^2 \left( \frac{a_{j1}}{b_j} \right)^2 - 1
$$

It follows that there is exactly one root with positive imaginary part and one with negative imaginary part since $a_{j1}$ is real and $\Im b_j \neq 0$. By inspection,

$$
Q_s \cap L_\infty = \left\{ [0, 1, i], [0, 1, -i] \right\}
$$

**\[\square\]**

Embed $\mathbb{C}^2$ in $\mathbb{P}^2$ by the standard map

$$
(z_1, z_2) \in \mathbb{C}^2 \longrightarrow [z_0, z_1, z_2] \in \mathbb{P}^2
$$
Let $\omega$ be the meromorphic two form on $\mathbb{P}^2$ given by

$$\omega = \frac{dz_1 \wedge dz_2}{\prod_{j=1}^n (a_jz_1 + a_jz_2 - b_j)}$$

If $n \geq 3$, then $\omega$ has simple poles on the lines $L_1 \cdots, L_n$ and is holomorphic on the difference $\mathbb{P}^2 \setminus \left(L_1 \cup L_2 \cup \cdots \cup L_n \right)$. If $n = 2$, then $\omega$ has simple poles on $L_1 L_2 L_\infty$, and is holomorphic on $\mathbb{P}^2 \setminus \left(L_1 \cup L_2 \cup L_\infty \right)$.

Regard the union of lines

$$C = L_1 \cup L_2 \cup \cdots \cup L_n \cup L_\infty$$

as a singular algebraic curve of degree $n + 1$ in $\mathbb{P}^2$. Then, the Poincare’ residue of $\omega$ along the curve $C$ (see, for example [GH], p.147),

$$\eta = \text{res}_C \omega$$

is a Rosenlicht differential on $C$ (see [S], Ch. IV.9).

Fix, $j = 1, \cdots, n$. For each $k \neq j$,

$$\omega = \frac{1}{\det \begin{pmatrix} a_{j1} & a_{j2} \\ a_{k1} & a_{k2} \end{pmatrix}} \prod_{\ell \neq j,k} \left( a_{\ell 1}z_1 + a_{\ell 2}z_2 - b_\ell \right) \frac{d(a_{j1}z_1 + a_{j2}z_2 - b_j)}{a_{j1}z_1 + a_{j2}z_2 - b_j} \wedge \frac{d(a_{k1}z_1 + a_{k2}z_2 - b_k)}{a_{k1}z_1 + a_{k2}z_2 - b_k}$$

It follows from this representation that the restriction of $\eta$ to the projective line $L_j$ has a simple pole at the point $d^{i,j}$ for all $k = 1, \cdots, n, k \neq j$, with residue

$$\frac{1}{\det \begin{pmatrix} a_{j1} & a_{j2} \\ a_{k1} & a_{k2} \end{pmatrix}} \prod_{\ell \neq j,k} \left( \frac{1}{d^{j,k}_0} \left( a_{\ell 1}d^{i,j}_1 + a_{\ell 2}d^{i,j}_2 - b_\ell d^{i,j}_0 \right) \right)^{-1}$$

Furthermore, it is holomorphic and does not vanish on $L_j \setminus \{ d^{j,\alpha} | \alpha = 1, \cdots, n, \infty, j \neq \alpha \}$. Consequently, $\eta |_{L_j}$ has a zero of order $n - 3$ at $d^{j,\infty}$, when $n \geq 3$, and a simple pole when $n = 2$. It follows from the preceding remarks that the restriction $\eta |_{L_j}$, has all the properties of the rational one form $\eta_j$ introduced above. In particular, $\eta_j$ exists and

$$\eta_j = \eta |_{L_j}$$
Set

\[ \eta_\infty = \eta|_{L_\infty} \]

For \( n \geq 3 \), the restriction \( \eta_\infty = 0 \), since \( \omega \) is regular along \( L_\infty \). If \( n = 2 \), then \( \eta_\infty \) has simple poles at the points \( d^{1,\infty} \) and \( d^{2,\infty} \), and no other poles.

We need the following residue formula.

**Lemma II.3** Fix \( \alpha = 1, \ldots, n, \infty \). Let \( p_1 \) and \( p_2 \) be any two points on

\[ L_\alpha \setminus \{ d^{\alpha,\beta} \mid \beta = 1, \ldots, n, \infty, \beta \neq \alpha \} \]

Suppose \( S_1 \) and \( S_2 \) are algebraic curves in \( \mathbb{P}^2 \) such that \( L_\alpha \) meets \( S_1 \) transversally at \( p_1 \) and \( S_2 \) transversally at \( p_2 \). Let

\[ \pi : T_\alpha \longrightarrow L_\alpha \setminus \{ d^{\alpha,\beta} \mid \beta = 1, \ldots, n, \infty, \beta \neq \alpha \} \]

be a tubular neighborhood of \( L_\alpha \setminus \{ d^{\alpha,\beta} \mid \beta = 1, \ldots, n, \infty, \beta \neq \alpha \} \) such that

\[ S_i \cap T_\alpha = \pi^{-1}(p_i) \]

for \( i = 1, 2 \). Finally, let \( \gamma \) be a path on \( L_\alpha \setminus \{ d^{\alpha,\beta} \mid \beta = 1, \ldots, n, \infty, \beta \neq \alpha \} \) joining \( p_1 \) to \( p_2 \) and set

\[ \Gamma = \pi^{-1}(\gamma) \cap \partial T \]

The “cylinder” \( \Gamma \) is oriented so that

\[ \partial \Gamma = \partial(S_2 \cap T_\alpha) - \partial(S_1 \cap T_\alpha) \]

Here, \( S_i \cap T_\alpha, i = 1, 2 \), is given the standard orientation of an open subset of the Riemann surface \( S_i \). Then,

\[ \int_\Gamma \omega = -2\pi i \int_\gamma \eta_\alpha \]

**Proof:** The proof is similar to that of [Seb].
Remark. By convention, the boundary of an oriented surface is oriented such that the ordered pair of the outward pointing normal and an oriented tangent vector gives the orientation of the surface.

Proof of Theorem II.1: By construction, for all $\alpha = 1, \ldots, n, \infty$, the projective line $L_\alpha$ intersects the quadric $Q_s$ transversally at the points $c_\alpha^\pm(s)$. For $0 \leq s \leq \rho$ let

$$\pi_{\alpha,s} : T_{\alpha,s} \rightarrow L_\alpha \setminus \{ d^{\alpha,\beta} \mid \beta = 1, \ldots, n, \infty, \beta \neq \alpha \}$$

be tubular neighborhoods of $L_\alpha \setminus \{ d^{\alpha,\beta} \mid \beta = 1, \ldots, n, \infty, \beta \neq \alpha \}$ such that

$$Q_t^\pm \cap T_{\alpha,s} = \pi_{\alpha,s}^{-1}(c_\alpha^\pm(t))$$

for $t = 0, s$, and

$$\Delta_{\alpha,0}^\pm = Q_0^\pm \cap T_{\alpha,s}$$

is independent of $s$. Set

$$\Delta_{\alpha,s}^\pm = Q_s^\pm \cap T_{\alpha,s}$$

By convention, the disks $\Delta_{\alpha,t}^\pm, t = 0, s$, are oriented as subsets of the Riemann surface $Q_t^\pm$. Also, set

$$\Gamma_{\alpha,s}^\pm = \pi_{\alpha,s}^{-1}\left( \{ c_\alpha^\pm(t) \mid 0 \leq t \leq s \} \right) \cap \partial T_{\alpha,s}$$

The “cylinder” $\Gamma_{\alpha,s}^\pm$ is oriented so that

$$\partial \Gamma_{\alpha,s}^\pm = \partial \Delta_{\alpha,s}^\pm - \partial \Delta_{\alpha,0}^\pm$$

By Lemma II.3,

$$\int_{\Gamma_{\alpha,s}^\pm} \omega = -2\pi i \int_{\{ c_\alpha^\pm(t) \mid 0 \leq t \leq s \}} \eta_\alpha$$

Now, for each $0 \leq s \leq \rho$, set

$$Z_s^\pm = \pm D_s(0) - \left( P_s^\pm - P_0^\pm + \sum_{j=1}^{n} \Gamma_{j,s}^\pm + \Gamma_{\infty,s}^\pm \right)$$
where, for \( t = 0, s \)

\[
\mathcal{P}_t^\pm = \mathcal{Q}_t^\pm \setminus \left( \bigcup_{j=1}^n \Delta_{j,t}^\pm \cup \Delta_{\infty,t}^\pm \right)
\]

We have for \( t = 0, s \)

\[
\partial \mathcal{P}_t^\pm = \partial \mathcal{Q}_t^\pm - \sum_{j=1}^n \partial \Delta_{j,t}^\pm - \partial \Delta_{\infty,t}^\pm
\]

and

\[
\partial \mathcal{Q}_s^\pm = \pm \partial \mathcal{D}_s(0), \quad \partial \mathcal{Q}_0^\pm = \emptyset
\]

Therefore,

\[
\partial Z_s^\pm = \pm \partial \mathcal{D}_s(0) - \left( \partial \mathcal{Q}_s^\pm - \sum_{j=1}^n \partial \Delta_{j,s}^\pm - \partial \Delta_{\infty,s}^\pm \right) + \left( \partial \mathcal{Q}_0^\pm - \sum_{j=1}^n \partial \Delta_{j,0}^\pm - \partial \Delta_{\infty,0}^\pm \right)
- \sum_{j=1}^n \left( \partial \Delta_{j,s}^\pm - \partial \Delta_{j,0}^\pm \right) - \left( \partial \Delta_{\infty,s}^\pm - \partial \Delta_{\infty,0}^\pm \right)
= 0
\]

In other words, \( Z_s^\pm \) is a 2-cycle in \( \mathbb{P}^2 \setminus C \).

Observe that the homology class \([Z_s^\pm] \in H_2(\mathbb{P}^2 \setminus C, \mathbb{Z})\) represented by \( Z_s^\pm \) is 0 since, by construction, \( Z_s^\pm \) depends continuously on \( s \) and \([Z_0^\pm] = 0\). The integral of the holomorphic two form \( \omega \) over the one dimensional complex manifolds \( \mathcal{P}_t^\pm, t = 0, s \) vanishes. It follows that

\[
0 = \int_{Z_s^\pm} \omega = \pm \int_{\mathcal{D}_s(0)} \omega - \sum_{j=1}^n \int_{\Gamma_{j,s}^\pm} \omega - \int_{\Gamma_{\infty,s}^\pm} \omega
= \pm \int_{\mathcal{D}_s(0)} \omega + 2\pi i \sum_{j=1}^n \int_{\{c_{j}^\pm(s)\cap 0 \leq s \leq \rho\}} \eta_j + 2\pi i \int_{\{c_{\infty}^\pm(s)\cap 0 \leq s \leq \rho\}} \eta_{\infty}
\]

Since, \( c_{\infty}^+(s) \) and \( c_{\infty}^-(s) \) do not depend on \( s \),

\[
\int_{\{c_{\infty}^\pm(s)\cap 0 \leq s \leq \rho\}} \eta_{\infty} = 0
\]

Therefore

\[
\int_{\mathcal{D}_s(q)} \omega = \mp 2\pi i \sum_{j=1}^n \int_{\{c_{j}^\pm(s)\cap 0 \leq s \leq \rho\}} \eta_j
\]
**Remark.** For all $0 \leq s \leq \rho$, the divisor of the rational function

$$\frac{z_1^2 + z_2^2 - s^2 z_0^2}{z_1^2 + z_2^2}$$

on the curve $C$ is

$$\sum_{j=1}^{n} \left( c_j^+(s) - c_j^+(0) \right) + \left( c_j^-(s) - c_j^-(0) \right)$$

It therefore also follows from Abel’s theorem for the singular curve $C$ and the Rosenlicht differential $\eta$ that

$$\sum_{j=1}^{n} \int_{\{c_j^+(s) \mid 0 \leq s \leq \rho\}} \eta_j = -\sum_{j=1}^{n} \int_{\{c_j^-(s) \mid 0 \leq s \leq \rho\}} \eta_j$$
§III The Evaluation of Fermion Loops

Let \( q_i = (q_{i0}, q_{i}) , i = 1, 2, \cdots , n + 1 \), be vectors in \( \mathbb{R} \times \mathbb{R}^2 \). We assume that no three of the points \( q_1, \cdots , q_{n+1} \) lie on a line. The amplitude of the \((n+1)\)-loop with momenta \( q_1, q_2, \cdots , q_{n+1} \) is

\[
I(q_1, \cdots , q_{n+1}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} dk_1 \, dk_2 \int_{-\infty}^{+\infty} \frac{dk_0}{\prod_{i=1}^{n+1} (i(k-q_i)_0 - e(k-q_i))}
\]

where the dispersion relation \( e(k) \) is given by

\[
e(k) = \frac{1}{2m} \left( k^2 - k_F^2 \right) \]

Here, \( k_F = \sqrt{2m\mu} \) is the Fermi momentum.

If \( e(k - q_i) \neq 0 \) for \( i = 1, \cdots , n + 1 \), and \( n \geq 1 \) then, by the residue theorem (closing the contour in the upper half plane),

\[
\int_{-\infty}^{+\infty} \frac{dk_0}{\prod_{i=1}^{n+1} (i(k_0-q_{i0})-e(k-q_i))} = 2\pi \, m^{n} \, \sum_{i=1}^{n+1} \frac{\chi(e(k-q_i)<0)}{\prod_{j=1}^{n+1} f_{ij}(k)}
\]

where

\[
f_{ij}(k) = m \left( e(k-q_i) - e(k-q_j) + i(q_{i0} - q_{j0}) \right) = (q_j - q_i) \cdot k + \frac{1}{2} \left( q_i^2 - q_j^2 \right) + im (q_{i0} - q_{j0})
\]
Observe that \( f_{ij}(k), \ i, j = 1, \cdots n + 1, \) is an affine linear function of \( k. \) Substituting, we obtain
\[
I(q_1, \cdots, q_{n+1}) = \frac{m^n}{(2\pi)^2} \sum_{i=1}^{n+1} \int_{|k-q_i|<k_F} \frac{dk_1 dk_2}{\prod_{j=1, j\neq i}^{n+1} f_{ij}(k)}
\]

It follows from this representation that \( I(q_1, \cdots, q_{n+1}) \) is a continuous function on the set
\[
\left\{ \left( (q_{i_0}, q_1), \cdots, (q_{n+1}, q_{n+1}) \right) \in \left( \mathbb{R} \times \mathbb{R}^2 \right)^{n+1} \left| q_{i_0} \neq q_{j_0} \text{ for } i \neq j \right. \right\}
\]

Observe that Theorem II.1 can be used to compute each of the summands
\[
\int_{|k-q_i|<k_F} \frac{dk_1 dk_2}{\prod_{j=1, j\neq i}^{n+1} f_{ij}(k)}
\]

For each pair \( 1 \leq i \neq j \leq n + 1, \)
\[
\ell_{ij} = \{ z \in \mathbb{C}^2 \mid f_{ij}(z) = 0 \}
\]
is a line in \( \mathbb{C}^2. \) We have
\[
\ell_{ij} = \ell_{ji}
\]
since \( f_{ij} = -f_{ji}. \) If \( q_{i_0} = q_{j_0} \) the line \( \ell_{ij} \cap \mathbb{R}^2 \) is the perpendicular bisector of the points \( q_i \) and \( q_j. \) Observe that for any three different indices \( 1 \leq i, j, k \leq n + 1 \) the lines \( \ell_{ij} \) and \( \ell_{jk} \) intersect in exactly one point \( d^{ijk} \) because \( q_i, q_j \) and \( q_k \) do not lie on a line. Furthermore, \( d^{ijk} \) is the common intersection point of the three lines \( \ell_{ij} = \ell_{ji}, \ell_{jk} = \ell_{kj}, \ell_{ki} = \ell_{ik}, \) since
\[
f_{ij} + f_{jk} + f_{ki} = 0
\]

Observe that \( \Re d^{ijk} \) is the center of the circle circumscribing the triangle with vertices \( q_i, q_j, q_k. \) If \( q_{i_0} = q_{i_0} = q_{i0}, \) then the point \( d^{ijk} \) is real.
For any three different indices $1 \leq i, j, k \leq n + 1$, put

$$r_{ijk} = \frac{1}{\det (q_j - q_i, q_k - q_i)} \frac{1}{\prod_{\nu \neq i, j, k}^n f_{i\nu}(d^{ijk})}$$

where $\det (q_j - q_i, q_k - q_i)$ is the determinant of the matrix with rows $q_j - q_i$ and $q_k - q_i$. Observe that $|\det (q_j - q_i, q_k - q_i)|$ is twice the area of the triangle with vertices $q_i, q_j, q_k$.

By construction,

$$0 = f_{ij}(d^{ijk}) = -f_{j\nu}(d^{ijk}) - f_{\nu i}(d^{ijk}) = -f_{j\nu}(d^{ijk}) + f_{i\nu}(d^{ijk})$$

so that

$$r_{ijk} = -r_{jik}$$

Let $l_{ij}, i, j = 1, \ldots, n + 1$, be the projective closure of $l_{ij}$ in the complex projective plane $\mathbb{P}^2$ and, as in the last section, let $L_\infty$ be the line at infinity. For each pair of indices $1 \leq i \neq j \leq n + 1$, let $\eta_{ij}$ be the unique meromorphic differential form on $l_{ij}$ that is holomorphic outside the points $d^{ijk}, k = 1, \ldots, n + 1, k \neq i, j$, and $l_{ij} \cap L_\infty$, that has a simple pole with residue $r_{ijk}$ at the point $d^{ijk}$, and that has a zero of order $n - 3$ (respectively, a simple pole for $n = 2$) at $l_{ij} \cap L_\infty$. By construction,

$$\eta_{ij} = -\eta_{ji}$$
If \( 1 \leq i \neq j \leq n + 1 \), \( q_{i0} \neq q_{j0} \) and \( 0 < s < k_F \), let \( c_{ij}^\pm (s) \) be the point where the line \( \ell_{ij} \) meets the quadric

\[
Q_i^\pm(s) = \{ z \in \mathbb{C}^2 \mid (z_1 - q_{i1})^2 + (z_2 - q_{i2})^2 = s^2 \text{ and } \pm \frac{z_2 - q_{j2}}{z_1 - q_{i1}} > 0 \}.
\]

In general \( c_{ij}^\pm (s) \neq c_{ji}^\pm (s) \).

**Theorem III.1** Let \( n \geq 2 \). Suppose that no three of the points \( q_1, \ldots, q_{n+1} \) in \( \mathbb{R}^2 \) lie on a line, that \( q_{i0} \neq q_{j0} \) for \( i \neq j \), and that \( d_{ijk} \neq d_{ijk} \) for \( k \neq k' \). Further, suppose that \( d_{ijk} \neq c_{ij}^\pm (s) \) for all \( i \neq j \) and all \( k \neq i, j \). Then,

\[
I(q_1, \ldots, q_{n+1}) = \frac{m}{2\pi} \sum_{i=j}^{n+1} \int_{\mathbb{C}} \left\{ c_{ij}^+ (s) \mid 0 < s < k_F \right\} \eta_{ij} = -\frac{m}{2\pi} \sum_{i=j}^{n+1} \int_{\mathbb{C}} \left\{ c_{ij}^- (s) \mid 0 < s < k_F \right\} \eta_{ij}
\]

**Proof:** By Theorem II.1,

\[
\int_{|k-q_i|<k_F} \frac{dk_1dk_2}{\prod_{j=1}^{n+1} f_{ij}(k)} = -2\pi \sum_{j \neq i} \int_{\mathbb{C}} \left\{ c_{ij}^+ (s) \mid 0 < s < k_F \right\} \eta_{ij} = +2\pi \sum_{j \neq i} \int_{\mathbb{C}} \left\{ c_{ij}^- (s) \mid 0 < s < k_F \right\} \eta_{ij}
\]

for \( i = 1, \ldots, n + 1 \).

**Remark.** For each pair \( 1 \leq i, j \leq n + 1 \), \( i \neq j \), choose \( k \neq i, j \). We parameterize the line \( \ell_{ij} \), by

\[
\pi_{ij}(z) = z \left( (q_j - q_i)_2, -(q_j - q_i)_1 \right) + d_{ijk}, \quad z \in \mathbb{C},
\]

The pull back \( \pi_{ij}^* \eta_{ij} \) of \( \eta_{ij} \) is a rational form on \( \mathbb{C} \) with simple poles. Consequently, there is a rational function \( \varphi_{ij}(z) \) such that

\[
\varphi_{ij} \, dz = \pi_{ij}^* \eta_{ij}
\]
We can therefore reformulate the statement of Theorem III.1 as

\[ I(q_1, \ldots, q_{n+1}) = \frac{m^n}{2\pi i} \sum_{i,j=1}^{n+1} \int_{w_{ij}} \varphi_{ij}(z) \, dz \]

where

\[ w_{ij} = \pi_{ij}^{-1} \left( \left\{ \varphi_{ij}(s) \mid 0 < s < k_F \right\} \right) \]

**Remark.** For \( n = 2 \), that is the 3-loop, we parameterize the line \( \ell_{ij} \), \( i, j = 1, 2, 3 \), \( i \neq j \), by

\[ \pi_{ij}(z) = z \left( (q_j - q_i)_2, -(q_j - q_i)_1 \right) + d \, , \, z \in \mathbb{C} \]

where \( d = d^{123} \) is the solution to the pair of equations

\[
\begin{align*}
(q_2 - q_1)k + \frac{1}{2} (q_1^2 - q_2^2) + z m(q_1_0 - q_2_0) &= 0 \\
(q_3 - q_1)k + \frac{1}{2} (q_1^2 - q_3^2) + z m(q_1_0 - q_3_0) &= 0
\end{align*}
\]

Observe that \( \pi_{ij}^* \eta_{ij} \) is a rational form on \( \mathbb{C} \) that has simple poles at 0 and \( \infty \) and no others. By construction,

\[
\pi_{ij}^* \eta_{ij} = \pm \frac{1}{\det (q_2 - q_1, q_3 - q_1)} \frac{dz}{z} = \pm \frac{1}{\det (q_2 - q_1, q_3 - q_1)} d \log z
\]

with \(+\) for \((i,j) = (1,2), (2,3), (3,1)\) and \(-\) for \((i,j) = (2,1), (3,2), (1,3)\). Let \( w_{ij}(s) \) be the unique root of

\[
(z(q_j - q_i)_2 + d_1 - q_i)_1^2 + (-z(q_j - q_i)_1 + d_2 - q_i)_2^2 = s^2
\]

with

\[
\Theta \frac{-w_{ij}(s)(q_j - q_i)_1 + d_2 - q_2}{w_{ij}(s)(q_j - q_i)_2 + d_1 - q_1} > 0
\]

We have

\[ I(q_1, q_2, q_3) = \frac{1}{2\pi i} \frac{m^2}{\det (q_2 - q_1, q_3 - q_1)} \sum_{i,j=1}^{3} \int_{w_{ij}} \left\{ w_{ij}(s) \mid 0 \leq s \leq k_F \right\} \pm d \log z \]

Clearly, each of the six integrals on the right hand side can be performed explicitly.
**Remark.** For each pair $1 \leq i, j \leq n+1$, $i \neq j$, and $k \neq i, j$ let $\eta_{ijk}$ be the unique meromorphic differential form on $\tilde{L}_{ij}$ that is holomorphic outside the points $d^{ijk}$, and $\tilde{L}_{ij} \cap L_\infty$, that has a simple pole with residue $r_{ijk}$ at the point $d^{ijk}$, and that has a simple pole at $\tilde{L}_{ij} \cap L_\infty$. Then

$$\eta_{ij} = \sum_{k \neq i, j} \eta_{ijk}$$

Consequently

$$I(q_1, \cdots, q_{n+1}) = \frac{m^n}{2\pi^4} \sum_{\text{pairs i,j are different}}^{n+1} \int \left\{ e_{ij}^+(s) \left| 0 < s < k_F \right. \right\} \eta_{ijk}$$

So $I(q_1, \cdots, q_{n+1})$ is decomposed into a sum of terms each of which only depends on a triple $\{q_i, q_j, q_k\}$ of momenta. The fact that such a decomposition is possible follows directly from the identity

$$\frac{1}{\prod_{i=1}^{n+1} (i(k-q_i) - e(k-q_i))} = \sum_{1 \leq i < j < k \leq n+1} \frac{1}{\prod_{i \neq j, k}^{n+1} \nu_{i,j,k} (d^{ijk}) (i(k-q_j) - e(k-q_j)) (i(k-q_k) - e(k-q_k))}$$

This identity was pointed out to us by A. Cavalli and P. Wagner.

Fix $p_1, \cdots, p_{n+1}$ in $\mathbb{R}^2$. We assume that no three of these points lie on a line and no four of them lie on a circle. The main purpose of this section is to show that the limit

$$J(p_1, \cdots, p_{n+1}) = \lim_{q_i \to (0, p_i)} \lim_{q_j \to (0, p_j)} I(q_1, \cdots, q_{n+1})$$

exists and to evaluate it. To do this, first set

$$F_{ij}(k) = m \left( e(k - p_i) - e(k - p_j) \right) = (p_j - p_i) \cdot k + \frac{1}{2} (p_i^2 - p_j^2)$$

for all $i, j = 1, \cdots, n+1$, and observe that

$$\lim_{q_i \to (0, p_i)} f_{ij}(k) = F_{ij}(k)$$
Furthermore, for each $1 \leq i \neq j \leq n+1$ the line $l_{ij}$ converges to the perpendicular bisector 

$$L_{ij} = \{ z \in \mathbb{C}^2 \mid F_{ij}(z) = 0 \}$$

of $\mathbf{p}_i$ and $\mathbf{p}_j$. Also, let $D^{ijk}$ be the center of the circle circumscribing the triangle with vertices $\mathbf{p}_i, \mathbf{p}_j$ and $\mathbf{p}_k$. Then,

$$\lim_{q_i \to (0, \mathbf{p}_i)} d^{ijk} = D^{ijk}$$

We have $D^{ijk} \neq D^{ijk'}$ for all $k \neq k'$, since no four of the points $\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}$ lie on a circle.

For any three different indices $1 \leq i, j, k \leq n+1$, let

$$R_{ijk} = \frac{1}{\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} \frac{1}{\prod_{\nu=1}^{n+1} F_{\nu}(D^{ijk})}$$

Then,

$$\lim_{q_i \to (0, \mathbf{p}_i)} r_{ijk} = R_{ijk}$$

Observe that

$$R_{ijk} = R_{jki} = R_{kij} = -R_{jik}$$

Let

$$\triangle(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) = \{ \mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k \}$$

be the triangle with vertices $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$. We can, using the antisymmetry of $R_{ijk}$, define

$$R_{\triangle} = \begin{cases} R_{ijk}, & \text{if } \mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k \text{ are oriented counter clockwise} \\ -R_{ijk}, & \text{if } \mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k \text{ are oriented clockwise} \end{cases}$$

for every triangle $\triangle = \triangle(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k)$. Let

$$\rho_{\triangle} = |D^{ijk} - \mathbf{p}_i| = |D^{ijk} - \mathbf{p}_j| = |D^{ijk} - \mathbf{p}_k|$$

be the radius of the circle circumscribing $\triangle$. Since

$$F_{\nu}(D^{ijk}) = m \left( e(D^{ijk} - \mathbf{p}_i) - e(D^{ijk} - \mathbf{p}_\nu) \right)$$

$$= \frac{1}{2} \left( \rho_{\triangle}^2 - |D^{ijk} - \mathbf{p}_\nu|^2 \right)$$

we have the geometric description of $R_{ijk}$

$$R_{ijk} = \frac{1}{\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} \frac{1}{\prod_{\nu=1}^{n+1} \rho_{\triangle}^2 - |D^{ijk} - \mathbf{p}_\nu|^2}$$

$$= \frac{2^{n-2}}{(n+1) \prod_{\nu=1}^{n+1} (\rho_{\triangle}^2 - |D^{ijk} - \mathbf{p}_\nu|^2)}$$
Theorem III.2 Let $n \geq 2$ and let $p_1, \ldots, p_{n+1}$ be points in $\mathbb{R}^2$ such that no three lie on a line and no four lie on a circle. Furthermore, assume that for every triple $1 \leq i, j, k \leq n + 1$ of pairwise distinct indices the triangle $\triangle = \triangle(p_i, p_j, p_k)$ is not a right triangle and $\rho_\triangle \neq k_F$. Let $T$ be the set of all triangles $\triangle$ with vertices in the set $\{p_1, \ldots, p_{n+1}\}$, and let $T_{ac}$ be the set of those triangles in $T$ that have no angle bigger than $90^\circ$, i.e. acute triangles. Then,

$$J(p_1, \ldots, p_{n+1}) = m^n \sum_{\substack{\triangle \in T_{ac} \\ \rho_\triangle > k_F}} R_\triangle - 2m^n \sum_{\substack{\triangle \in T \\ s_{\text{edge of } \triangle} = \rho_\triangle \leq |s| > 2k_F}} \epsilon_\triangle,s R_\triangle \text{ arccot} \sqrt{\frac{4\rho_\triangle^2 - |s|^2}{|s|^2 - 4k_F^2}}$$

Here, for an edge $s$ of a triangle $\triangle$ we denote by $|s|$ the length of $s$ and set

$$\epsilon_\triangle,s = \begin{cases} 
+1 & \text{if the angle of } \triangle \text{ at the vertex opposite to } s \text{ is acute} \\
-1 & \text{if the angle of } \triangle \text{ at the vertex opposite to } s \text{ is obtuse}
\end{cases}$$

Corollary III.3

i) If $|p_i - p_j| < 2k_F$ for all $i, j = 1, \ldots, n + 1$, then

$$J(p_1, \ldots, p_{n+1}) = m^n \sum_{\substack{\triangle \in T_{ac} \\ \rho_\triangle > k_F}} R_\triangle$$

If, in particular, $k_F < k'_F$ such that $k_F > \frac{1}{2} \min_{k=1,\ldots,n+1} |p_i - p_j|$ and such that there is no triangle $\triangle \in T_{ac}$ with $k_F < \rho_\triangle < k'_F$, then

$$J(p_1, \ldots, p_{n+1}; k_F) = J(p_1, \ldots, p_{n+1}; k'_F)$$

ii) If all $n + 1$ disks with radius $k_F$ around the points $p_1, \ldots, p_{n+1}$ have at least one point in common, then

$$J(p_1, \ldots, p_{n+1}) = 0$$

(iii) Suppose $n = 2$ and $|p_i - p_j| < 2k_F$ for all $i, j = 1, 2, 3$. Let $\triangle = \triangle(p_1, p_2, p_3)$. Then,

$$J(p_1, p_2, p_3) = \begin{cases} 
0, & \text{if the disks of radius } k_F \text{ around } p_1, p_2, p_3 \\
\frac{m^2}{2 \times \text{area of } \triangle}, & \text{otherwise}
\end{cases}$$
Proof of the Corollary: Part (i) follows immediately from Theorem III.2. If \( \Delta = \Delta(p_1, p_2, p_3) \) is a triangle with only acute angles and the disks around the points \( p_1, p_2, p_3 \) with radius \( k_F \) have at least one point in common, then \( \rho_\Delta \leq k_F \). So, part (ii) follows from part (i). Finally, part (iii) is a special case of (i) and (ii).

The rest of this section is devoted to the proof of Theorem III.2. For each pair of indices \( 1 \leq i \neq j \leq n + 1 \), let \( \omega_{ij} \) be the unique meromorphic differential form on \( \overline{L}_{ij} (= \text{projective closure of} \ L_{ij}) \) that is holomorphic outside the points \( D^{ijk} \), \( k = 1, \cdots, n + 1, k \neq i, j \), and \( L_{ij} \cap L_\infty \), that has a simple pole with residue \( R_{ijk} \) at the point \( D^{ijk} \), and that has a zero of order \( n - 3 \) (respectively, a simple pole for \( n = 2 \)) at \( L_{ij} \cap L_\infty \). By construction,

\[ \omega_{ij} = -\omega_{ji} \]

If \( 0 \leq k_F \leq \frac{1}{2} |p_i - p_j| \), let \( C_{ij}^{\pm} \) be the intersection of the line \( L_{ij} \) with

\[ Q_{ij}^{\pm}(k_F) = \left\{ z \in \mathbb{C}^2 \mid (z_1 - p_{i1})^2 + (z_2 - p_{i2})^2 = k_F^2 \right\} \]

Observe that \( C_{ij}^{+} = C_{ji}^{-} \). It is useful to introduce the notation

\[ p = \left( (0, p_1), \cdots, (0, p_{n+1}) \right) \in \left( \mathbb{R} \times \mathbb{R}^2 \right)^{n+1} \]

and

\[ \mathcal{M} = \left\{ q = \left( (q_{i1}, q_1), \cdots, (q_{n+1}, q_{n+1}) \right) \in \left( \mathbb{R} \times \mathbb{R}^2 \right)^{n+1} \mid q_i \neq q_j \text{ for } i \neq j \right\} \]

Now, by Theorem III.1,

\[ J(p_1, \cdots, p_{n+1}) = \lim_{q \to p, q \in \mathcal{M}} m_{2\pi i} \sum_{i \neq j}^{n+1} \int_{s^+} \omega_{ij} \eta_{ij} \]

It is easy to see that

\[ \lim_{q \to p, q \in \mathcal{M}} \eta_{ij} = \omega_{ij} \]

pointwise. However, it takes some work to determine the limiting behavior

\[ \lim_{q \to p, q \in \mathcal{M}} \left\{ c_{ij}^+(s) \mid 0 < s < k_F \right\} \]
of the path \( \left\{ c_{ij}^{+}(s) \mid 0 < s < k_F \right\} \). There are two cases, \( 0 \leq s < \frac{1}{2} |p_i - p_j| \) and \( k_F \geq s \geq \frac{1}{2} |p_i - p_j| \).

Suppose \( 0 \leq s < \frac{1}{2} |p_i - p_j| \) for some \( 1 \leq i, j \leq n + 1 \). Then, the line \( L_{ij} \) intersects

\[
Q_{ij}^\pm(s) = \left\{ z \in \mathbb{C}^2 \mid (z_1 - p_{i1})^2 + (z_2 - p_{i2})^2 = s^2 \quad \text{and} \quad \Im \frac{z_2 - p_{i2}}{z_1 - p_{i1}} > 0 \right\}
\]

in exactly one point which is denoted by \( C_{ij}^\pm(s) \). Again, \( C_{ij}^+(s) = C_{ji}^-(s) \) and

\[
C_{ij}^\pm(s) = \lim_{\frac{q \to p}{q \in \mathbb{P}}} C_{ij}^\pm(s)
\]

Also,

\[
\Re C_{ij}^\pm(s) = \frac{1}{2}(p_i + p_j)
\]

Suppose that \( s > \frac{1}{2} |p_i - p_j| \). Then,

\[
L_{ij} \cap \left\{ x \in \mathbb{R}^2 \mid |x - p_i| = s \right\} = L_{ij} \cap \left\{ x \in \mathbb{R}^2 \mid |x - p_j| = s \right\}
\]

consists of two points which we again denote by \( C_{ij}^\pm(s) \). By convention, \( C_{ij}^+(s) \) is the point that makes the triangle with vertices \( p_i, p_j \) and \( C_{ij}^+(s) \) counter clockwise oriented, and \( C_{ij}^-(s) \) is the point that makes the triangle with vertices \( p_i, p_j \) and \( C_{ij}^-(s) \) clockwise oriented.
Lemma III.4 Let $1 \leq i, j \leq n + 1$.

(i) Suppose that $0 \leq s < \frac{1}{2} |p_i - p_j|$. Let $x \neq \frac{1}{2}(p_i + p_j)$ in $L_{jk} \cap \mathbb{R}^2$ be a real point on the perpendicular bisector of $p_i$ and $p_j$. Then, the triangle in the complex line $L_{ij}$ with the vertices $C_{ij}^+(s)$, $C_{ij}^-(s)$ and $x$ has the same orientation as the triangle in $\mathbb{R}^2$ with the vertices $p_i$, $p_j$ and $x$.

(ii) Suppose that $s > \frac{1}{2} |p_i - p_j|$. Then, $C_{ij}^\pm(s) = C_{ji}^\mp(s)$ and

$$C_{ij}^\pm(s) = \lim_{q \to p, q \in M \atop q_i = 0} c_{ij}^\pm(s) = \lim_{q \to p, q \in M \atop q_i = 0, q_j > 0} c_{ij}^\pm(s) = \lim_{q \to p, q \in M \atop q_i = 0, q_j < 0} c_{ij}^\pm(s) = \lim_{q \to p, q \in M \atop q_i = 0, q_j = 0} c_{ij}^\pm(s)$$

Proof: For convenience, we set $m = 1$. We may assume, by applying an orientation preserving isometry, that

$$p_i = (-a, 0) \quad \text{and} \quad p_j = (a, 0)$$

with $a > 0$. Now,

$$L_{ij} = \{ (0, z_2) \mid z_2 \in \mathbb{C} \}$$

and

$$Q_i^\pm(s) = \{ z \in \mathbb{C}^2 \mid (z_1 + a)^2 + z_2^2 = s^2 \text{ and } \pm \Im \frac{z_2}{z_1 + a} > 0 \}$$
If \( s < a \), then the first statement of Lemma III.4 follows from

\[
C_{ij}^\pm(s) = (0, \pm i \sqrt{a^2 - s^2})
\]

If, on the other hand, \( s > a \),

\[
C_{ij}^\pm(s) = (0, \pm \sqrt{s^2 - a^2}) = C_{ji}^\mp(s)
\]

Suppose \( q \in \mathcal{M} \) is close to \( p \). Then, by a \( q \)-dependent orientation preserving isometry close to the identity, we may assume that

\[
q_i = (-a', 0) \quad \text{and} \quad q_j = (a', 0)
\]

with \( a' > 0 \) close to \( a \). Now,

\[
\ell_{ij} = \left\{ z \in \Phi^2 \mid z_1 = -i \frac{q_i - q_j}{2a'} \right\}
\]

and the second statement of Lemma III.4 follows directly from

\[
c_{ij}^\pm(s) = \left(-i \frac{q_i - q_j}{2a'}, \pm \text{sgn} (q_i - q_j) \sqrt{s^2 - a'^2 + \frac{1}{4a'^2} (q_i - q_j)^2 + i(q_i - q_j)} \right)
\]

\[
c_{ji}^\pm(s) = \left(-i \frac{q_j - q_i}{2a'}, \pm \text{sgn} (q_j - q_i) \sqrt{s^2 - a'^2 + \frac{1}{4a'^2} (q_j - q_i)^2 - i(q_i - q_j)} \right)
\]

where we make a branch cut for the square root along the negative real axis.

If \( 0 \leq k_F < \frac{1}{2} |p^{(j)} - p^{(k)}| \), then, by construction,

\[
\lim_{\substack{q \in \mathcal{M} \\ q \to p}} \{ c_{ij}^\pm(s) \mid 0 \leq s \leq k_F \} = \{ C_{ij}^\pm(s) \mid 0 \leq s \leq k_F \}
\]

is the straight line segment \([C_{ij}^\pm(0), C_{ij}^\pm]\) joining the point \( C_{ij}^\pm(0) \) to the point \( C_{ij}^\pm = C_{ij}^\pm(k_F) \) in \( L_{ij} \).
If \( k_F > \frac{1}{2}|p^{(j)} - p^{(k)}| \), then by Lemma III.4 (ii),

\[
\lim_{q \to p, \pm(q_0 - q_{j0}) > 0} \left\{ c_{ij}^+(s) \mid 0 \leq s \leq k_F \right\} \cup \left\{ c_{ij}^-(s) \mid k_F \geq s \geq 0 \right\}
\]

is the path obtained by composing the three paths:

1. The straight segment in \( L_{ij} \) joining \( C_{ij}^+(0) \) to \( \frac{1}{2}(p_i + p_j) \)
2. The segment on \( L_{ij} \cap \mathbb{R}^2 \) joining \( \frac{1}{2}(p_i + p_j) \) to \( C_{ij}^+(k_F) \), followed by the inverse of this segment

and

3. The straight segment in \( L_{ij} \) joining \( \frac{1}{2}(p_i + p_j) \) to \( C_{ij}^-(0) \).
We encounter two problems in calculating

\[
\lim_{q \to p, \pm (q_{i0} - q_{j0}) > 0} \left( \int \{ c_{ij}^+(s) \mid 0 < s < k_F \} \eta_{ij} + \int \{ c_{ji}^+(s) \mid 0 < s < k_F \} \eta_{ji} \right)
\]

\[
= \lim_{q \to p, \pm (q_{i0} - q_{j0}) > 0} \left( \int \{ c_{ij}^+(s) \mid 0 \leq s \leq k_F \} \cup \{ c_{ji}^+(s) \mid k_F \geq s \geq 0 \} \right) \eta_{ij}
\]

First, the path (2) depends on the sign of \( q_{i0} - q_{j0} \). Second, the path (2) may contain one or more of the points \( D_{ij}^{jkl} \), that are the poles of the limiting meromorphic form \( \omega_{ij} \).

Let \( q \in \mathcal{M} \) be close to \( p \). For all indices \( 1 \leq j \neq k \leq n + 1 \) with \( k_F > \frac{1}{2} |p_i - p_j| \), let \([c_{ij}^+(k_F), c_{ji}^+(k_F)]\) be the straight line segment in \( l_{ij} \) joining \( c_{ij}^+(k_F) \) to \( c_{ji}^+(k_F) \). The composition of the paths

\[
\{ c_{ij}^+(s) \mid 0 \leq s \leq k_F \}
\]

\([c_{ij}^+(k_F), c_{ji}^+(k_F)]\)

and

\[
\{ c_{ji}^+(s) \mid k_F \geq s \geq 0 \}
\]

is a path joining \( c_{ij}^+(0) \) to \( c_{ji}^+(0) \).
For any three different indices $1 \leq i, j, k \leq n + 1$, recall that $\mathbb{R}d_{ijk}$ is the center of the circle circumscribing the triangle with the vertices $q_i, q_j, q_k$. Let

$$\alpha_{ijk} = \frac{1}{2} \det(q_i - \mathbb{R}d_{ijk}, q_j - \mathbb{R}d_{ijk})$$

be the oriented area of the triangle with vertices $q_i, q_j, \mathbb{R}d_{ijk}$, and

$$A_{ijk} = \frac{1}{2} \det(q_i - q_k, q_j - q_k)$$

the oriented area of the full triangle $q_i, q_j, q_k$.

For all $1 \leq i \neq j \leq n + 1$, set

$$\mathcal{I}_{ij} = \left\{ 1 \leq k \leq n + 1 \left| |D_{ijk} - p_i| < k_F \right. \right\}$$

$$= \left\{ 1 \leq k \leq n + 1 \left| \rho\triangle_{p_i,p_j,p_k} < k_F \right. \right\}$$

$$\mathcal{I}_{ij}^+(q) = \left\{ k \in \mathcal{I}_{ij} \left| q_{i0} - q_{j0} > 0 \text{ and the triangle with vertices } p_i, p_j, D_{ijk} \text{ is counterclockwise oriented} \right. \right\}$$

$$\mathcal{I}_{ij}^-(q) = \left\{ k \in \mathcal{I}_{ij} \left| q_{i0} - q_{j0} < 0 \text{ and the triangle with vertices } p_i, p_j, D_{ijk} \text{ is clockwise oriented} \right. \right\}$$

Finally, recall that for any three different indices $1 \leq i, j, k \leq n + 1$, $r_{ijk}$ is the residue of $\eta_{ij}$ at $d_{ijk}$, and if $\triangle = \triangle(q_i, q_j, q_k)$ is the triangle with vertices $q_i, q_j, q_k$, set

$$r_{\triangle} = \begin{cases} r_{ijk}, & \text{if } q_i, q_j, q_k \text{ are oriented counterclockwise} \\ -r_{ijk}, & \text{if } q_i, q_j, q_k \text{ are oriented clockwise} \end{cases}$$
Lemma III.5 Let \( q \in \mathcal{M} \) be close to \( p \) and suppose that for all pairwise distinct indices \( 1 \leq i, j, k \leq n + 1 \),

\[
\frac{\alpha_{ijk}}{A_{ijk}} \frac{q_k - q_i}{q_i - q_j} \neq \frac{\alpha_{kij}}{A_{kij}}
\]

(i) Fix \( 1 \leq i \neq j \leq n + 1 \) with \( k_F > \frac{1}{2} |p_i - p_j| \). Then,

\[
\int_{\{c^+_{ij}(s)\} |0 \leq s \leq k_F\}} \eta_{ij} + \int_{\{c^+_{ji}(k_F)\}, c^+_{ji}(k_F)]} \eta_{ij} + \int_{\{c^+_{ji}(s)\} |0 \leq s \leq k_F\}} \eta_{ji}
\]

\[
= \int_{c^+_{ij}(0)}^{c^+_{ij}(0)} \eta_{ij} - 2\pi i \sum_{k=1}^{n+1} \chi_{ijk} r_{ijk}
\]

Here, \( \int_{c^+_{ij}(0)} \eta_{ij} \) is the integral along the straight line segment \( [c^+_{ij}(0), c^+_{ji}(0)] \) in \( \ell_{ij} \), and

\[
\chi_{ijk} = \begin{cases} 
0 & \text{unless } k \in \mathcal{I}_{ij}^+(q) \cup \mathcal{I}_{ij}^-(q) \text{ and } \frac{\alpha_{ij}}{A_{ijk}} \frac{q_k - q_i}{q_i - q_j} < \frac{\alpha_{kij}}{A_{kij}} \text{ and } \frac{\alpha_{ij}}{A_{ijk}} \frac{q_j - q_k}{q_i - q_j} < \frac{\alpha_{ijk}}{A_{ijk}} \\
\pm 1 & \text{if } k \in \mathcal{I}_{ij}^\pm(q) \text{ and } \frac{\alpha_{ij}}{A_{ijk}} \frac{q_k - q_i}{q_i - q_j} < \frac{\alpha_{kij}}{A_{kij}} \text{ and } \frac{\alpha_{ij}}{A_{ijk}} \frac{q_j - q_k}{q_i - q_j} < \frac{\alpha_{ijk}}{A_{ijk}} 
\end{cases}
\]

(ii) Let \( 1 \leq i, j, k \leq n + 1 \) be any three different indices and let \( \Delta = \Delta(q_i, q_j, q_k) \). Then,

\[
\sum_{\{i', j', k'\} \neq \{i,j,k\}} \chi_{i'j'k'} r_{i'j'k'} = \begin{cases} 
2r_\Delta & \text{if } \rho_\Delta < k_F \text{ and all angles in the triangle } \Delta \text{ are smaller than } 90^\circ \\
0 & \text{otherwise}
\end{cases}
\]

Lemma III.5 is now used to obtain an intermediate expression for the function \( J(p_1, \ldots, p_{n+1}) \) appearing in Theorem III.2. Recall that \( p_1, \ldots, p_{n+1} \) are points in \( \mathbb{R}^2 \) such that no three of them lie on a line and no four of them lie on a circle, and furthermore, that for every triple \( 1 \leq i, j, k \leq n + 1 \) of pairwise distinct indices the triangle \( \Delta = \Delta(p_i, p_j, p_k) \) is not a right triangle and \( \rho_\Delta \neq k_F \). Here, as above, \( \rho_\Delta \) is the radius of the circle circumscribing the triangle \( \Delta \).

Proposition III.6

\[
J(p_1, \ldots, p_{n+1}) = \frac{m}{4\pi} \sum_{1 \leq i \neq j \leq n + 1} \int_{C^+_{ij}(0)}^0 \omega_{ij} + \frac{m}{4\pi} \sum_{1 \leq i \neq j \leq n + 1} \int_{C^+_{ij}(0)}^0 \omega_{ij} - m \sum_{\Delta \in \mathcal{I}_{n+1}} R_\Delta
\]

\[
\sum_{\rho_\Delta < k_F} R_\Delta
\]

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Proof: By Theorem III.1 and Lemma III.5 (i),

\[
\frac{2}{m} I(q_1, \cdots, q_{n+1}) = \frac{1}{2\pi i} \sum_{1 \leq i \neq j \leq n+1} \left( \int_{c_{ij}^+(s) | 0 < s < k_F} \eta_{ij} + \int_{c_{ij}^+(s) | 0 < s < k_F} \eta_{ji} \right)
\]

\[
= \frac{1}{2\pi i} \sum_{1 \leq i \neq j \leq n+1} \left( \int_{c_{ij}^+(k_F)} \eta_{ij} + \int_{c_{ij}^+(k_F)} \eta_{ji} \right)
\]

\[
+ \frac{1}{2\pi i} \sum_{1 \leq i \neq j \leq n+1} \left( \int_{c_{ij}^+(0)} \eta_{ij} - \int_{c_{ij}^+(k_F)} \eta_{ij} \right) + \sum_{k=1}^{n+1} \chi_{ijn} r_{ijk}
\]

on an open, dense subset of \( \mathcal{M} \). By Lemma III.5 (ii),

\[
\frac{2}{m} I(q_1, \cdots, q_{n+1}) = \frac{1}{2\pi i} \sum_{1 \leq i \neq j \leq n+1} \left( \int_{c_{ij}^+(k_F)} \eta_{ij} + \int_{c_{ij}^+(k_F)} \eta_{ji} \right)
\]

\[
+ \frac{1}{2\pi i} \sum_{1 \leq i \neq j \leq n+1} \left( \int_{c_{ij}^+(0)} \eta_{ij} - \int_{c_{ij}^+(k_F)} \eta_{ij} \right) - 2 \sum_{\Delta \in \mathcal{T}_{ac} \ 0 < k_F} r_{\Delta}
\]

on an open, dense subset of \( \mathcal{M} \). In fact the last identity holds for all \( q \) in \( \mathcal{M} \) near \( p \), since both sides are continuous functions on \( \mathcal{M} \) near \( p \). Finally, by Lemma III.4,

\[
\frac{2}{m^n} J(p_1, \cdots, p_{n+1}) = 2 \lim_{q \in \mathcal{M} \atop q \to p} I(q_1, \cdots, q_{n+1})
\]

\[
= \frac{1}{2\pi i} \sum_{1 \leq i \neq j \leq n+1} \left( \int_{C_{ij}^+(k_F)} \omega_{ij} - \int_{C_{ij}^+(k_F)} \omega_{ji} \right)
\]

\[
+ \frac{1}{2\pi i} \sum_{1 \leq i \neq j \leq n+1} \left( \int_{C_{ij}^+(0)} \omega_{ij} - \int_{C_{ij}^+(k_F)} \omega_{ij} \right) - 2 \sum_{\Delta \in \mathcal{T}_{ac} \ 0 < k_F} R_{\Delta}
\]

\[
= \frac{1}{2\pi i} \sum_{1 \leq i \neq j \leq n+1} \left( \int_{C_{ij}^+(0)} \omega_{ij} - \int_{C_{ij}^+(k_F)} \omega_{ij} \right) + \sum_{\Delta \in \mathcal{T}_{ac} \ 0 < k_F} R_{\Delta}
\]
The Proof of Lemma III.5 : For convenience, we again set \( m = 1 \). To prove part (i), let \( \gamma_{ij}(q) \) be the closed path obtained by composing \( \{ c_{ij}^+(s) \mid 0 \leq s \leq k_F \}, \{ c_{ij}^+(k_F), c_{ij}^+(k_F) \}, \{ c_{ij}^+(s) \mid k_F \geq s \geq 0 \} \) and \( [c_{ij}^+(0), c_{ij}^+(0)] \). We have
\[
\int_{\{ c_{ij}^+(s) \mid 0 \leq s \leq k_F \}} \eta_{kj} + \int_{[c_{ij}^+(k_F), c_{ij}^+(k_F)]} \eta_{kj} + \int_{\{ c_{ij}^+(s) \mid k_F \geq s \geq 0 \}} \eta_{ji} = \int_{c_{ij}^+(0)} \eta_{ij} + \int_{\gamma_{ij}(q)} \eta_{ij}
\]
since, \( \eta_{ij} = -\eta_{ji} \). By the residue theorem
\[
\int_{\gamma_{ij}(q)} \eta_{ij} = 2\pi i \sum_{k=0}^{n+1} \left( \text{winding number of } \gamma_{ij} \text{ around } d^{ijk} \right) \times \text{res} \eta_{ij}(d^{ijk})
\]
We shall show that the winding number of \( \gamma_{ij}(q) \) around \( d^{ijk} \) is \(-\chi_{ijk}\).

Fix \( 1 \leq k \leq n + 1, \ k \neq i, j \). A necessary condition that the winding number of \( \gamma_{ij}(q) \) around \( d^{ijk} \) is not zero is that \( D^{ijk} \) be a point on the limiting path (see, Figure 6)
\[
\lim_{q' \to q} \gamma_{ij}(q') = \gamma_{ij}(q)
\]
By Lemma III.4 (ii), our criterion becomes
\[
D^{ijk} \in \left\{ \left[ \frac{1}{2}(p_i + p_j), C_{ij}^+(k_F) \right], \ q_{i0} - q_{j0} > 0 \right\}
\]
\[
\left[ \frac{1}{2}(p_i + p_j), C_{ij}^-(k_F) \right], \ q_{i0} - q_{j0} < 0 \right\}
\]
since, \( D^{ijk} \) is real. Observe that, by construction, the distance between \( D^{ijk} \) and the midpoint \( \frac{1}{2}(p_i + p_j) \) is smaller than the distance between \( C_{ij}^+(k_F) \) and \( \frac{1}{2}(p_i + p_j) \) if and only if \( |D^{ijk} - p_i| < k_F \). Also, observe that the point \( D^{ijk} \) lies on the same side of \( \frac{1}{2}(p_i + p_j) \) as \( C_{ij}^+(k_F) \) if and only if the triangle with vertices \( p_i, p_j, D^{ijk} \) is counter clockwise oriented, and on the same side of \( \frac{1}{2}(p_i + p_j) \) as \( C_{ij}^-(k_F) \) if and only if the triangle with vertices \( p_i, p_j, D^{ijk} \) is clockwise oriented. Thus, the winding number of \( \gamma_{ij}(q) \) around \( d^{ijk} \) is zero when \( k \notin I_{ij}^+(q) \cup I_{ij}^-(q) \).
Now, suppose that $k \in \mathcal{I}^+_\ell (q) \cup \mathcal{I}^-_{ij} (q)$. As in the proof of Lemma III.4 we may assume that

$$p_i = (-a, 0) \quad \text{and} \quad p_j = (a, 0)$$

with $a > 0$ and

$$q_i = (-a', 0) \quad \text{and} \quad q_j = (a', 0)$$

with $a'$ close to $a$. We have

$$d_{ijk} = \left( -\frac{q_i - q_j}{2a'} , \frac{1}{2q_{k2}} \left( |q_k|^2 - a' \right) + \frac{1}{q_{k2}} \left( q_k + \frac{q_{k1} - a'}{2a'} \left( q_{i0} - \frac{q_{k1} + a'}{2a'} - q_{j0} \right) \right) \right)$$

The function

$$g(t) = \left( -\frac{q_i - q_j}{2a'} , \frac{1}{2q_{k2}} \left( |q_k|^2 - a' \right) + \frac{1}{q_{k2}} \right)$$

defined for $t \in \mathbb{R}$ parameterizes the real line in $\ell_{ij}$ passing through $d_{ijk}$ parallel to the imaginary $\Re z_2$-axis. By construction $g(t_0) = d_{ijk}$, where

$$t_0 = \frac{1}{q_{k2}} \left( q_{k0} + \frac{q_{k1} - a'}{2a'} - q_{i0} - \frac{q_{k1} + a'}{2a'} - q_{j0} \right)$$

It is easy to see that

$$t_i = \frac{q_{k2}}{|q_k|^2 - a'^2} (q_{i0} - q_{j0})$$

is the only real number for which there exists an $s > 0$ such that $g(t) \in Q^+ (s)$. Similarly,

$$t_j = -\frac{q_{k2}}{|q_k|^2 - a'^2} (q_{i0} - q_{j0})$$

is the only real number for which there exists an $s > 0$ such that $g(t) \in Q^+ (s)$. Our hypothesis (see, the statement of Theorem III.2) implies that $q_i$, $q_j$ and $q_k$ are not the vertices of a right triangle. By Thales’ theorem (see, Euclid, Book III, §31), the denominator $|q_k|^2 - a'^2$ does not vanish and

$$\frac{q_{k2}}{|q_k|^2 - a'^2} > 0$$

when $k \in \mathcal{I}^+_\ell (q)$ and

$$\frac{q_{k2}}{|q_k|^2 - a'^2} < 0$$

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when $k \in I_{ij}^-(q)$. Consequently,

$$t_j < 0 < t_i$$

since $k \in I_{ij}^+(q) \cup I_{ij}^-(q)$.

The line $g(t), -\infty < t < \infty$, meets the loop $\gamma_{ij}$ in exactly the two points $g(t_i)$ and $g(t_j)$. Furthermore, the winding number of $\gamma_{ij}$ around $d^{ijk} = g(t_0)$ is

$$\begin{cases} 
0 & \text{when } t_i < t_0 \text{ or } t_0 < t_j \\
\mp 1 & \text{if } t_j < t_0 < t_i \text{ and } k \in I_{ij}^+(q) 
\end{cases}$$

The first case is immediate from the figure.

If, on the other hand, $t_j < t_0 < t_i$, then the winding number is $-1$ when the triangle with vertices $c_{ij}^+(0), c_{ji}^+(0), d^{ijk}$ is counter clockwise oriented. This occurs, if and only if the limiting triangle with vertices $C_{ij}^+(0), C_{ji}^-(0) = C_{ji}^+(0), D^{ijk}$ is counter clockwise oriented. By Lemma III.4 (i), this happens if and only if the triangle with vertices $p_i, p_j, D^{ijk}$ is counter clockwise oriented. That is, $k \in I_{ij}^+(q)$. The case of winding number $+1$ is similar.

The last step in the proof of part (i) is to express the conditions on $t_i$, $t_0$ and $t_j$ that determine the winding number of $\gamma_{ij}$ around $d^{ijk}$ in terms of the quantities $\alpha_{ijk}$ and $A_{ijk}$. To start with,

$$t_0 = \frac{1}{\alpha_{ijk}} \left( q_{k0} + \frac{q_{k1} - a'}{2a'} q_{i0} - \frac{q_{k1} + a'}{2a'} q_{j0} \right) < \frac{q_{k2}}{|q_k|^2 - a'^2} (q_{i0} - q_{j0}) = t_i$$
is equivalent to

\[ \frac{\alpha_{ijk}}{A_{ijk}} \frac{1}{q_{i0} - q_{j0}} \left( q_{i0} + \frac{q_{k1} - a'}{2a'} - q_{i0} - \frac{q_{k1} + a'}{2a'} q_{j0} \right) < \frac{1}{2} \]

since,

\[ \alpha_{ijk} = \frac{a'}{2q_{k2}} \left( |q_{k1}|^2 - a'^2 \right) \]

\[ A_{ijk} = a' q_{k2} \]

and \( t_i > 0 \). This inequality is in turn equivalent to

\[ \frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{k0} - q_{i0}}{q_{i0} - q_{j0}} < \frac{1}{2} - \frac{\alpha_{ijk}}{A_{ijk}} \frac{1}{q_{i0} - q_{j0}} \left( q_{i0} + \frac{q_{k1} - a'}{2a'} - q_{i0} - \frac{q_{k1} + a'}{2a'} q_{j0} \right) \]

\[ = \frac{1}{2} \left( 1 - \frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{k1} + a'}{a'} \right) \]

We will show that

\[ \frac{1}{2} \left( 1 - \frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{k1} + a'}{a'} \right) = \frac{\alpha_{kij}}{A_{kij}} \]

In other words,

\[ t_0 < t_i \Leftrightarrow \frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{k0} - q_{i0}}{q_{i0} - q_{j0}} < \frac{\alpha_{kij}}{A_{kij}} \]

Let \( 2\epsilon_{\ell} > 0, \ell = i, j, k \), be the length of the edge opposite to the vertex \( q_{\ell} \) in the triangle with vertices \( q_i, q_j, q_k \), and let \( h_{\ell} \) be the (signed) height of the triangle over the edge opposite to the vertex \( q_{\ell} \). Clearly,

\[ A_{ijk} = e_{\ell} \cdot h_{\ell} \]

for \( \ell = i, j, k \). We also let \( s_{\ell} \) be the (signed) height of \( \Re d_{ijk} \) over the edge opposite to the vertex \( q_{\ell} \). Clearly,

\[ \alpha_{ijk} = e_k \cdot s_k = a' \cdot s_k \]

\[ \alpha_{jk} = e_i \cdot s_i \]

\[ \alpha_{ki} = e_j \cdot s_j \]

Finally, let \( b \) be the point where the perpendicular bisector of \( q_i, q_j \) meets the line through \( q_i \) and \( q_k \).

The two triangles in the figure below with vertex \( b \) and solid edges are similar.
Therefore,  
\[ \frac{s_j}{\frac{1}{2}h_j} = \frac{b_2 - s_k}{b_2} = 1 - \frac{s_k}{b_2} \]

Also, the triangle with vertices \( \mathbf{q}_i = (-a', 0), 0, \mathbf{b} \) is similar to the triangle with vertices \( \mathbf{q}_i = (-a', 0), (\mathbf{q}_{k1}, 0), \mathbf{q}_k \). Therefore,

\[ \frac{\mathbf{q}_{k2}}{b_2} = \frac{a' + \mathbf{q}_{k1}}{a'} \]

It follows that

\[ \frac{\alpha_{kkj}}{\alpha_{kij}} = \frac{s_j}{h_j} = \frac{1}{2} \left( 1 - \frac{s_k}{b_2} \right) = \frac{1}{2} \left( 1 - \frac{s_k}{\mathbf{q}_{k2}} \frac{a' + \mathbf{q}_{k1}}{a'} \right) = \frac{1}{2} \left( 1 - \frac{\alpha_{ijk}}{\alpha_{ijk}} \frac{a' + \mathbf{q}_{k1}}{a'} \right) \]

as claimed above.

By the same sort of argument,

\[ t_j < t_0 \iff \frac{\alpha_{ijk}}{\alpha_{ijk}} \frac{q_{i0} - q_{k0}}{q_{i0} - q_{j0}} < \frac{\alpha_{jki}}{\alpha_{jki}} \]

The proof of the first part of Lemma III.5 is now complete.

(ii) If \( k_F < \rho_\Delta \), then, by definition, \( \chi^{i'j'k'} = 0 \) for all permutations \( i', j', k' \) of \( i, j, k \). We may therefore assume that \( \rho_\Delta < k_F \). We may further assume that the triangle with vertices
\( q_i, q_j, q_k \) is counter clockwise oriented since, by direct inspection, \( \chi_{ijk} = -\chi_{jik} \) and, as observed before, \( r_{ijk} = -r_{jik} \). It thus suffices to show that

\[
\chi_{ijk} + \chi_{jki} + \chi_{kij} = \begin{cases} 
1, & \text{if } \triangle \text{ is an acute triangle} \\
0, & \text{if } \triangle \text{ is an obtuse triangle} 
\end{cases}
\]

Clearly, one or two of the differences

\[
q_{i0} - q_{j0}, \ q_{j0} - q_{k0}, \ q_{k0} - q_{i0}
\]

is positive, since \( (q_{i0} - q_{j0}) + (q_{j0} - q_{k0}) + (q_{k0} - q_{i0}) = 0 \). We first verify the statement of the last paragraph in the case that \( \triangle(q_i, q_j, q_k) \) is an acute triangle whose vertices \( q_i, q_j, q_k \) are counter clockwise oriented and exactly one of the differences is positive. Observe that \( 0 < A_{ijk} = A_{jki} = A_{kij} \) since \( q_i, q_j, q_k \) are counter clockwise oriented and that \( \alpha_{ijk}, \alpha_{jki}, \alpha_{kij} > 0 \) since \( \triangle(q_i, q_j, q_k) \) is acute.

We may assume, without loss of generality, that

\[
q_{i0} - q_{j0} > 0, \ q_{j0} - q_{k0} < 0, \ q_{k0} - q_{i0} < 0
\]

By construction, \( i \notin I_{jk}^+(q) \cup I_{jk}^-(q) \) and \( j \notin I_{ki}^+(q) \cup I_{ki}^-(q) \) and therefore, by definition,

\[
\chi_{jki} = \chi_{kij} = 0
\]

On the other hand, \( k \in I_{ij}^+(q) \) and

\[
\frac{\alpha_{ijk} q_{i0} - q_{j0}}{A_{ijk}} < 0 < \frac{\alpha_{kij} q_{i0} - q_{j0}}{A_{kij}}
\]

\[
\frac{\alpha_{ijk} q_{j0} - q_{k0}}{A_{ijk}} < 0 < \frac{\alpha_{jki} q_{j0} - q_{k0}}{A_{jki}}
\]

so that, by definition,

\[
\chi_{ijk} = 1
\]

We next treat the case in which \( \triangle(q_i, q_j, q_k) \) is an acute triangle whose vertices \( q_i, q_j, q_k \) are counter clockwise oriented and exactly two of the differences are positive. We assume, without loss of generality, that

\[
q_{i0} - q_{j0} > 0, \ q_{j0} - q_{k0} > 0, \ q_{k0} - q_{i0} < 0
\]
As above, the third inequality implies \( \chi_{kij} = 0 \). On the other hand, the first two inequalities imply \( k \in T^+_i(q), \ i \in T^+_j(q) \) and consequently,

\[
\chi_{ijk} = \begin{cases} 
1 & \text{if } \frac{a_{ijk} q_i - q_j}{A_{ijk} q_i - q_j} < \frac{a_{jki}}{A_{jki}} \\
0 & \text{otherwise}
\end{cases}
\]

\[
\chi_{jki} = \begin{cases} 
1 & \text{if } \frac{a_{jki} q_i - q_j}{A_{jki} q_i - q_j} < \frac{a_{ijk}}{A_{ijk}} \\
0 & \text{otherwise}
\end{cases}
\]

This shows that

\[
\chi_{ijk} + \chi_{jki} = 1
\]

The proof that

\[
\chi_{ijk} + \chi_{jki} + \chi_{kij} = 1
\]

when \( \Delta \) is an acute, counter clockwise oriented triangle is now complete.

The case of an obtuse triangle is similar. \( \blacksquare \)

The proof of Theorem III.2 requires two ingredients in addition to Proposition III.6.

**Proposition III.7** For each pair \( 1 \leq i, j \leq n + 1, \ i \neq j \), and \( k \neq i, j \) let \( \omega_{ijk} \) be the unique meromorphic differential form on \( T_{ij} \) that is holomorphic outside the points \( D_{ijk} \), and \( T_{ij} \cap L_{\infty} \), that has a simple pole with residue 1 at the point \( D_{ijk} \), and that has a simple pole at \( T_{ij} \cap L_{\infty} \). If \( |p_i - p_j| > 2k_F \) then

\[
\int_{C_{ij}^{+}}^{C_{ij}^{-}} \omega_{ijk} = -4\pi i \epsilon_{\Delta,(p_i,p_j)} R_{\Delta} \arccot \sqrt{\frac{4\rho_{\Delta}^2 - |p_i - p_j|^2}{|p_i - p_j|^2 - 4k_F^2}}
\]

where \( \Delta = \Delta(p_i, p_j, p_k) \) and

\[
\epsilon_{\Delta,(p_i,p_j)} = \begin{cases} 
+1 & \text{if the angle of } \Delta \text{ at } p_k \text{ is acute} \\
-1 & \text{if the angle of } \Delta \text{ at } p_k \text{ is obtuse}
\end{cases}
\]

**Proposition III.8**

\[
\sum_{1 \leq i \neq j \leq n+1} \int_{C_{ij}^{+}(0)}^{C_{ij}^{-}(0)} \omega_{ij} = 4\pi i \sum_{\Delta \in T_{\infty}} R_{\Delta}
\]

Before proving Proposition III.7 and Proposition III.8, we give the
Proof of Theorem III.2: By Proposition III.6 and Proposition III.8,

\[
\frac{1}{m} J(p_1, \cdots, p_{n+1}) = \frac{1}{4\pi} \sum_{1 \leq i \neq j \leq n+1} \int_{C_{ij}^-} \omega_{ij} + \frac{1}{4\pi} \sum_{1 \leq i \neq j \leq n+1} \frac{1}{|p_i - p_j| > 2k_F} \int_{C_{ij}^+} \omega_{ij} - \sum_{\Delta \in T_{nc}} R_{\Delta} \]

\[
= \sum_{\Delta \in T_{nc}} R_{\Delta} + \frac{1}{4\pi} \sum_{1 \leq i \neq j \leq n+1} \int_{|p_i - p_j| > 2k_F} \omega_{ij} - \sum_{\Delta \in T_{nc}} R_{\Delta} \]

\[
= \frac{1}{4\pi} \sum_{1 \leq i \neq j \leq n+1} \int_{|p_i - p_j| > 2k_F} \omega_{ij} + \sum_{\Delta \in T_{nc}} R_{\Delta} \]

By construction for \(1 \leq i, j \leq n+1, i \neq j\)

\[
\omega_{ij} = \sum_{k \neq i, j} R_{ijk} \omega_{ijk} \]

Therefore by Proposition III.7

\[
\frac{1}{4\pi} \sum_{1 \leq i \neq j \leq n+1} \int_{|p_i - p_j| > 2k_F} \omega_{ij} \]

\[
= -\sum_{1 \leq i \neq j \leq n+1} \sum_{1 \leq k \leq n+1} \epsilon_{\Delta(p_i,p_j,p_k), (p_i,p_j)} R_{\Delta} \Delta(p_i,p_j,p_k) \arccot \sqrt{\frac{4\rho_\Delta^2 - |p_i - p_j|^2}{|p_i - p_j|^2 - 4k_F^2}} \]

\[
= -2 \sum_{\Delta \in T_{nc}} \epsilon_{\Delta} R_{\Delta} \arccot \sqrt{\frac{4\rho_{\Delta}^2 - |s|^2}{|s|^2 - 4k_F^2}} \]

Proof of Proposition III.7: By possibly interchanging \(i\) and \(j\) we may assume that the triangle with vertices \(p_i, p_j, p_k\) is counterclockwise oriented. As in the proof of Lemma III.4 we may assume that \(p_i = (-a, 0)\) and \(p_j = (a, 0)\) with \(a > 0\). Then there is \(b \neq 0\) such that \(D_{ijk} = (0, b)\). Then

\[
a^2 + b^2 = \rho_{\Delta}^2 \quad b = \epsilon_{\Delta, (p_i, p_j)} |b| \]

\[
L_{ij} = \{ (z_1, z_2) \in \mathbb{R}^2 \mid z_1 = 0 \} \quad C_{ij}^\pm = (0, \pm t \sqrt{a^2 - k_F^2}) \]

\[
\omega_{ijk} = \frac{d_{z_2}}{z_2 - b} \]

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Consequently
\[
\int_{C_{ij}^{+}}^{C_{ij}^{-}} \omega_{ijk} = \int_{b}^{a} \frac{dz}{\sqrt{a^2 - k_p^2}} = \int_{a}^{b} \frac{dz}{\sqrt{a^2 - k_p^2}} = -4\pi \operatorname{arccot} \frac{b}{\sqrt{a^2 - k_p^2}} = -4\pi \ell_{\Delta, (p_i, p_j)} \operatorname{arccot} \sqrt{\frac{\rho_p^2 - a^2}{a^2 - k_p^2}}
\]

The proof of Proposition III.8 requires a lemma. To prepare for Lemma III.9 pick a generic vector \((v_1, v_2) \in \mathbb{R}^2\) and, for each \(i = 1, \cdots, n + 1\), let
\[
g_i = \{ z \in \mathbb{C}^2 \mid v_1 z_1 + v_2 z_2 = v_1 p_{i1} + v_2 p_{i2} \}
\]
be the line through \(p_i\) whose slope is determined by \((v_1, v_2)\). For all pairs \(i \neq j\), let \(E_{ij}\) be the point where the lines \(g_i\) and \(L_{ij}\) meet. Furthermore, let \(\omega_{i\infty}\) be the rational differential form on the line \(L_{\infty}\) at infinity that is holomorphic outside the points of \(L_{\infty} \cap T_{ij}\) and has a pole of order one at \(L_{\infty} \cap T_{ij}\) with residue \(- \sum_{k \neq i, j} R_{ijk}\). Finally, let \(\gamma_{\infty}\) be the union of the paths
\[
\{ [0, (1-t)i-2v_2, 1-t+tv_1] \mid 0 \leq t \leq 1 \}
\]
\[
\{ [0, -(1-t)i-2v_2, 1-t+tv_1] \mid 1 \geq t \geq 0 \}
\]
lying in \(L_{\infty}\).

**Lemma III.9** For all \(1 \leq i \leq n + 1\),
\[
\sum_{j \neq i} \int_{C_{ij}^{+}(0)}^{C_{ij}^{-}(0)} \omega_{ij} = \int_{\gamma_{\infty}} \omega_{i\infty} + 2\pi \ell \sum_{j, k \neq i, j} \beta_{ijk} R_{ijk}
\]

where
\[
\beta_{ijk} = \begin{cases} 
+1 & \text{if } D_{ijk} \text{ lies between } \frac{1}{2}(p_i + p_j) \text{ and } E_{ij}, \\
-1 & \text{if } D_{ijk} \text{ lies between } \frac{1}{2}(p_i + p_j) \text{ and } E_{ij}, \\
0 & \text{otherwise}
\end{cases}
\]

**Proof:** Put
\[
F_t^{\pm}(z_1, z_2) = (1 - t) \left( (z_1 - p_{i1}) \pm i(z_2 - p_{i2}) \right) + t \left( v_1 z_1 + v_2 z_2 - v_1 p_{i1} - v_2 p_{i2} \right)
\]

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and let $E_{ij}^\pm(t)$ be the root of $F_{ij}^\pm$ on $L_{ij}$. Clearly, $E_{ij}^\pm(1) = E_{ij}$ and

$$g_i = \{ (z_1, z_2) \in \mathbb{C}^2 \mid F_{ij}^\pm(z_1, z_2) = 0 \}$$

$$Q_i^\pm(0) = \{ (z_1, z_2) \in \mathbb{C}^2 \mid F_{ij}^\pm(z_1, z_2) = 0 \}$$

and consequently,

$$\delta_{ij} = [C_{ij}^+(0), C_{ij}^-(0)] \cup \{ E_{ij}^-(t) \mid 0 \leq t \leq 1 \} \cup \{ E_{ij}^+(t) \mid 1 \geq t \geq 0 \}$$

is a loop that meets $L_{ij} \cap \mathbb{R}^2$ only in the points \( \frac{1}{2}(p_i + p_j) \) and $E_{ij}$.

Observe that the projective closure of $\{ (z_1, z_2) \in \mathbb{C}^2 \mid F_{ij}^\pm(z_1, z_2) = 0 \}$ meets the line $L_{\infty}$ at the point

$$E_{i\infty}(t) = [0, \mp(1-t)v_2, 1-tv_1, 1-t+tv_1]$$

We have

$$\gamma_{\infty} = \{ E_{i\infty}^+(t) \mid 0 \leq t \leq 1 \} \cup \{ E_{i\infty}^-(t) \mid 1 \geq t \geq 1 \}$$

for all $i = 1, \ldots, n + 1$.

The union of lines

$$X_i = L_{\infty} \cup \bigcup_{j \neq i} \overline{T_{ij}}$$

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is a singular curve in $\mathbb{P}^2$. The forms $\omega_{ij}, j = 1, \cdots, n + 1, \infty, j \neq i$, define a Rosenlicht differential on $X_i$. For each $t \in [0, 1]$ the divisor of the meromorphic function

$$\frac{E^-_t \cdot F^+_t}{F^-_0 \cdot F^+_0}$$

on the curve $X_i$ is

$$\sum_{j \neq i} \left( (E^-_{ij}(t) - E^-_{ij}(0)) + (E^+_{ij}(t) - E^+_{ij}(0)) \right)$$

By Abel's Theorem ([S], Ch. V.10, Proposition 5),

$$\sum_{j \neq i} \left( \int_{E^-_{ij}(t) \mid 0 \leq t \leq s} \omega_{ij} + \int_{E^+_{ij}(t) \mid s \geq t \geq 0} \omega_{ij} \right)$$

is independent of $s$. For $s = 0$ this quantity is zero. Therefore,

$$\sum_{j \neq i} \left( \int_{E^-_{ij}(t) \mid 0 \leq t \leq 1} \cup \{ E^+_{ij}(t) \mid 1 \geq t \geq 0 \} \omega_{ij} = 0 \right.$$

By the definition of the loop $\delta_{ij}$ and the conclusion of the last paragraph,

$$\sum_{j = 1}^{n+1} \int_{C^-_{ij}(0)} \omega_{ij} = \sum_{j = 1}^{n+1} \int_{C^+_{ij}(0)} \omega_{ij} + \int_{C_{i \infty}} \omega_{i \infty}$$

$$- \sum_{j \neq i} \left( \int_{E^-_{ij}(t) \mid 0 \leq t \leq 1} \cup \{ E^+_{ij}(t) \mid 1 \geq t \geq 0 \} \omega_{ij} \right.$$}

$$= 2 \pi t \sum_{j, k \neq i, j \neq k} \beta_{ijk} R_{ijk} + \int_{C_{i \infty}} \omega_{i \infty}$$

since, by Lemma III.4 (i), the winding number of $\delta_{ij}$ around $D^{ijk}$ is equal to $\beta_{ijk}$. \[\square\]

Finally, we give the

**Proof of Proposition III.8:** Observe that

$$\sum_{i=1}^{n+1} \omega_{i \infty} = 0$$

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since, by construction, the residues at each pole add up to zero. Now, by Lemma III.9,

\[
\sum_{1 \leq j \neq i \leq n+1} \int_{C_{ij}^+(0)} \omega_{ij} = \int_{\gamma_{\infty}} \sum_{i=1}^{n+1} \omega_{i\infty} + 2\pi \sum_{\text{pairwise different}} \beta_{ijk} R_{ijk} = 2\pi \sum_{\text{pairwise different}} \beta_{ijk} R_{ijk}
\]

Fix three pairwise different indices \( i, j, k \) and let \( \Delta = \Delta(p_i, p_j, p_k) \). Let

\[
h = \{ (x_1, x_2) \in \mathbb{R}^2 \mid v_1 x_1 + v_2 x_2 = v_1 D_{ij}^{jk} + v_2 D_{jk}^{ij} \}
\]

be the real line through the center \( D_{ijk} \) of the circle circumscribing \( \Delta \) whose slope is determined by \( (v_1, v_2) \). Then,

\[
\beta_{ijk} = \begin{cases} 
0 & \text{unless } h \text{ meets the segment between } p_i \text{ and } \frac{1}{2}(p_i + p_j) \\
\pm 1 & \text{if } h \text{ meets the segment between } p_i \text{ and } \frac{1}{2}(p_i + p_j) \\
& \text{and the triangle } p_i, p_j, D_{ijk} \text{ is counterclockwise (clockwise) oriented}
\end{cases}
\]

In particular, if \( \Delta \) is acute, there are exactly two permutations \( i', k', j' \) of \( i, j, k \) for which \( \beta_{i'k'j'} \neq 0 \).

Furthermore, if \( \beta_{i'k'j'} \neq 0 \), then

\[
\beta_{i'k'j'} = \pm 1 \iff p_{i'}, p_{j'}, p_{k'} \text{ counterclockwise (clockwise) oriented}
\]

\[
R_{i'k'j'} = \pm R_{\Delta}
\]
Consequently,
\[ \sum_{\{i', j', k'\} = \{i,j,k\}} \beta_{i'j'k'} R_{i'j'k'} = 2R \Delta \]
Similarly, if \( \Delta \) is obtuse,
\[ \sum_{\{i', j', k'\} = \{i,j,k\}} \beta_{i'j'k'} R_{i'j'k'} = 0 \]
Appendix

In this appendix we assume that \( J(p_1, p_2, p_3) \) exists and give a direct proof of

\[
J(p_1, p_2, p_3) = 0
\]

when the circle circumscribing the triangle with vertices \( p_1, p_2, p_3 \) has radius less than \( k_F \).

If the triangle is acute, the previous condition is equivalent to the statement that the three open disks \( K_1, K_2, K_3 \) with radius \( k_F \) around the points \( p_1, p_2, p_3 \) have a point in common. By translation invariance, the center of the circle circumscribing the triangle can be placed at the origin.

By definition,

\[
J(p_1, p_2, p_3) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} dk_1 \, dk_2 \int_{-\infty}^{+\infty} \frac{dk_0}{3 \prod_{i=1}^{3} \left( \frac{k_0 - e(k-p_i)}{k_0 + e(k-p_i)} \right)}
\]

The integral

\[
\int_{-\infty}^{+\infty} \frac{dk_0}{3 \prod_{i=1}^{3} \left( \frac{k_0 - e(k-p_i)}{k_0 + e(k-p_i)} \right)}
\]

is evaluated by closing the contour in the upper half plane when the point \( k \in \mathbb{R}^2 \) belongs to at most one of the disks \( K_1, K_2, K_3 \) and closing in the lower half plane when it belongs to at least two of the disks. In particular, the integral vanishes when \( k \notin K_1 \cup K_2 \cup K_3 \) or \( k \in K_1 \cap K_2 \cap K_3 \). We obtain

\[
\frac{(2\pi)^2}{m^2} J(p_1, p_2, p_3) = \int_{K_1 \backslash (K_2 \cup K_3)} \frac{dk_1 \, dk_2}{F_{12}(k) \, F_{13}(k)} - \int_{(K_2 \cup K_3) \backslash K_1} \frac{dk_1 \, dk_2}{F_{12}(k) \, F_{13}(k)}
\]

\[
+ \int_{K_3 \backslash (K_1 \cup K_2)} \frac{dk_1 \, dk_2}{F_{31}(k) \, F_{32}(k)} - \int_{(K_1 \cup K_2) \backslash K_3} \frac{dk_1 \, dk_2}{F_{31}(k) \, F_{32}(k)}
\]

\[
+ \int_{K_2 \backslash (K_3 \cup K_1)} \frac{dk_1 \, dk_2}{F_{23}(k) \, F_{21}(k)} - \int_{(K_3 \cup K_1) \backslash K_2} \frac{dk_1 \, dk_2}{F_{23}(k) \, F_{21}(k)}
\]

where, for all \( i, j = 1, 2, 3 \),

\[
F_{ij}(k) = m \left( e(k - p_i) - e(k - p_j) \right) = (p_j - p_i) \cdot k
\]

since, \( p_1^2 = p_2^2 = p_3^2 \). 

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Let $2r$ be the length of the secant to the circle $\partial K_1$ through $0$ that is perpendicular to the line through $0$ and $p_1$. Then the map
\[
\phi : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^2 \setminus \{0\}
\]
\[k \mapsto -\frac{r^2}{|k|^2} \; k\]
maps each of the circles $\partial K_1, \partial K_2, \partial K_3$ to itself ([Be], 10.8). By hypothesis, $0 \in K_i, i = 1, 2, 3$, and therefore, $\phi$ maps $K_i$ to the exterior of $\partial K_i$ and conversely. In particular,
\[
\phi \left( K_1 \setminus (\overline{K_2} \cup \overline{K_3}) \right) = (K_2 \cap K_3) \setminus \overline{K_1}
\]
Substituting,
\[
F_{12}(\phi(k)) \; F_{13}(\phi(k)) = \frac{r^4}{|k|^4} \; F_{12}(k) \; F_{13}(k)
\]
Also,
\[
\phi^* (dk_1 \wedge dk_2) = \frac{r^4}{|k|^4} \; dk_1 \wedge dk_2
\]
The last three equations imply that
\[
\int_{K_1 \setminus (K_2 \cup K_3)} \frac{dk_1 \; dk_2}{F_{12}(k) \; F_{13}(k)} = \int_{(K_2 \cap K_3) \setminus K_1} \frac{dk_1 \; dk_2}{F_{12}(k) \; F_{13}(k)} = 0
\]
The other pairs also cancel.
References


