The Weierstrass Function

Fix $\beta, \gamma \in (0, \infty)$. Then

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z} \\ \omega \neq 0}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} : \mathbb{C} \to \mathbb{C}$$

is the Weierstrass function with primitive periods γ, $i\beta$.

It obeys

a) $\wp(z)$ is analytic on $\mathbb{C} \setminus (\gamma\mathbb{Z} \oplus i\beta\mathbb{Z})$.

b) $\wp(z+\zeta) = \wp(z)$ for all $\zeta \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$.

c) $\wp(-z) = \wp(z)$ and $\overline{\wp(z)} = \wp(\bar{z})$

$\Rightarrow \wp(x + in\frac{\beta}{2}), \wp(iy + n\frac{\gamma}{2})$ are real $\forall x, y \in \mathbb{R}$, $n \in \mathbb{Z}$

d) Let $c \in \mathbb{C}$. Then $\wp(z) = c$ for exactly two z's in each fundamental domain.

$\Rightarrow \wp(z) = \wp(z')$ if and only if $z - z' \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$ or $z + z' \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$.

If $z \notin \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$ but $2z \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$, $\wp'(z) = 0$.

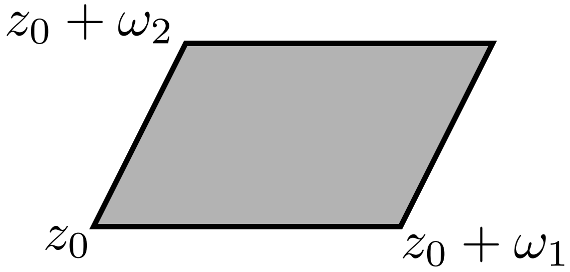
Theorem W.2 *Let $f(z)$ be a nonconstant meromorphic function that is periodic with respect to $\Omega = \omega_1\mathbb{Z}+\omega_2\mathbb{Z}$. Suppose that $f(z)$ has poles of order $n_1, \cdots, n_k$ at $p_1+\Omega, \cdots, p_k+\Omega$ and is analytic elsewhere. Let c be any complex number. Suppose that $f(z)-c$ has zeroes of order $m_1, \cdots, m_h$ at $z_1+\Omega, \cdots, z_h+\Omega$ and is nonzero elsewhere. Then*

$$\sum_{i=1}^{h} m_i = \sum_{i=1}^{k} n_k$$

Idea of Proof. For any nonconstant meromorphic function $f(z)$ and any domain $\mathcal{D}$

$$\int_{\partial\mathcal{D}} \tfrac{f'(z)}{f(z)}\,dz = 2\pi i\big[\#\text{ zeroes in }\mathcal{D} - \#\text{ poles in }\mathcal{D}\big]$$

Choose $\mathcal{D}$ of the form



with no zeroes or poles on $\mathcal{D}$.

Weierstrass Function Relatives

Define

$$\zeta(z) = \tfrac{1}{z} + \sum_{\substack{\omega\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}\\ \omega\neq 0}} \tfrac{1}{z-\omega} + \tfrac{1}{\omega} + \tfrac{z}{\omega^2}$$

$$\sigma(z) = z \prod_{\substack{\omega\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}\\ \omega\neq 0}} \left(1 - \tfrac{z}{\omega}\right) e^{\frac{z}{\omega}+\frac{1}{2}\frac{z^2}{\omega^2}}$$

They obey

a) $\zeta(z)$ is analytic on $\mathbb{C}\setminus(\gamma\mathbb{Z}\oplus i\beta\mathbb{Z})$. $\sigma(z)$ is entire and vanishes if and only if $z\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}$.

b) $\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$ and $\wp(z) = -\zeta'(z)$.

c) There are constants $\eta_1\in\mathbb{R}$, $\eta_2\in i\mathbb{R}$ satisfying $\eta_1 i\beta - \eta_2\gamma \ = \ 2\pi i$ such that

$$\zeta(z+\gamma) = \zeta(z) + \eta_1 \qquad \sigma(z+i\beta) = -\sigma(z)\, e^{\eta_2(z+i\frac{\beta}{2})}$$

$$\zeta(z+i\beta) = \zeta(z) + \eta_2 \qquad \sigma(z+\gamma) = -\sigma(z)\, e^{\eta_1(z+\frac{\gamma}{2})}$$

d) $\zeta(-z) = -\zeta(z)$ and $\overline{\zeta(z)} = \zeta(\bar{z})$.

$\Rightarrow \zeta(x)\in\mathbb{R}\ \forall x\in\mathbb{R}$ and $\zeta(iy)\in i\mathbb{R}\ \forall y\in\mathbb{R}$

$\sigma(-z) = -\sigma(z)$ and $\overline{\sigma(z)} = \sigma(\bar{z})$.

e) $\wp(u+v)+\wp(u)+\wp(v)=\big[\zeta(u+v)-\zeta(u)-\zeta(v)\big]^2$

Idea of Proof. For each fixed $v \in \mathbb{C} \setminus (\gamma\mathbb{Z} \oplus i\beta\mathbb{Z})$, both the left and right hand sides are periodic and have double poles, with the same singular part, at each $u \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$ and each $u \in -v+\gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$.

Define

$$k(z) = -i\big(\zeta(z) - z\tfrac{\eta_1}{\gamma}\big)$$

It obeys

a) $k(z)$ is analytic on $\mathbb{C} \setminus (\gamma\mathbb{Z} \oplus i\beta\mathbb{Z})$.

b) $k(z+\gamma) = k(z)$ and $k(z+i\beta) = k(z) - \frac{2\pi}{\gamma}$.

c) $k(-z) = -k(z)$ and $\overline{k(z)} = -k(\bar{z})$.

$\Rightarrow k(iy),\ k\big(iy+\frac{\gamma}{2}\big) \in \mathbb{R}$ for all $y \in \mathbb{R}$.

$k(x) \in i\mathbb{R},\ k(x+i\frac{\beta}{2}) \in \frac{\pi}{\gamma} + i\mathbb{R}$ for all $x \in \mathbb{R}$.

Set, for $z \in \mathbb{C} \setminus \big(\gamma\mathbb{Z} \oplus i\beta\mathbb{Z}\big)$,

$$\varphi(z,x) \;=\; e^{\zeta(z)x}\; \frac{\sigma\big(z - x - i\frac{\beta}{2}\big)}{\sigma\big(x + i\frac{\beta}{2}\big)}$$

$$\lambda(z) = -\wp(z)$$

$$k(z) = -i\big(\zeta(z) - z\tfrac{\eta_1}{\gamma}\big)$$

$$\xi(z) \;=\; e^{\gamma i k(z)} \;=\; e^{\gamma\zeta(z) - z\eta_1}$$

Lemma S.11

a) $-\frac{d^2}{dx^2}\varphi(z,x) \;+\; 2\wp\big(x + i\frac{\beta}{2}\big)\varphi(z,x) \;=\; \lambda(z)\varphi(z,x)$

b) $\varphi(z, x+\gamma) \;=\; \xi(z)\;\varphi(z,x)$

c) $\xi(z+\gamma) \;=\; \xi(z) \qquad\qquad \xi(z+i\beta) \;=\; \xi(z)$

Proof: a) First observe that

$$\frac{d}{dx}\frac{\sigma\left(z-x-i\frac{\beta}{2}\right)}{\sigma\left(x+i\frac{\beta}{2}\right)}$$

$$= -\left[\frac{\sigma'\left(z-x-i\frac{\beta}{2}\right)}{\sigma\left(z-x-i\frac{\beta}{2}\right)} + \frac{\sigma'\left(x+i\frac{\beta}{2}\right)}{\sigma\left(x+i\frac{\beta}{2}\right)}\right]\frac{\sigma\left(z-x-i\frac{\beta}{2}\right)}{\sigma\left(x+i\frac{\beta}{2}\right)}$$

$$= -\left[\zeta\left(z-x-i\tfrac{\beta}{2}\right) + \zeta\left(x+i\tfrac{\beta}{2}\right)\right]\frac{\sigma\left(z-x-i\frac{\beta}{2}\right)}{\sigma\left(x+i\frac{\beta}{2}\right)}$$

$$\Rightarrow \tfrac{d}{dx}\varphi(z,x) = \Big(\zeta(z) - \zeta\left(z-x-i\tfrac{\beta}{2}\right) - \zeta\left(x+i\tfrac{\beta}{2}\right)\Big)\,\varphi(z,x)$$

As $\left[\zeta(u+v)-\zeta(u)-\zeta(v)\right]^2 = \wp(u+v)+\wp(u)+\wp(v)$

$$\tfrac{d^2}{dx^2}\varphi(z,x)$$

$$= \Big(\zeta'\left(z-x-i\tfrac{\beta}{2}\right) - \zeta'\left(x+i\tfrac{\beta}{2}\right)\Big)\varphi(z,x)$$

$$+ \left[\zeta(z) - \zeta\left(z-x-i\tfrac{\beta}{2}\right) - \zeta\left(x+i\tfrac{\beta}{2}\right)\right]^2\varphi(z,x)$$

$$= -\Big(\wp\left(z-x-i\tfrac{\beta}{2}\right) - \wp\left(x+i\tfrac{\beta}{2}\right)\Big)\varphi(z,x)$$

$$+ \left[\zeta(z) - \zeta\left(z-x-i\tfrac{\beta}{2}\right) - \zeta\left(x+i\tfrac{\beta}{2}\right)\right]^2\varphi(z,x)$$

$$= -\Big(\wp\left(z-x-i\tfrac{\beta}{2}\right) - \wp\left(x+i\tfrac{\beta}{2}\right)\Big)\varphi(z,x)$$

$$+ \Big(\wp(z) + \wp\left(z-x-i\tfrac{\beta}{2}\right) + \wp\left(x+i\tfrac{\beta}{2}\right)\Big)\varphi(z,x)$$

$$= \Big(\wp(z) + 2\wp\left(x+i\tfrac{\beta}{2}\right)\Big)\varphi(z,x)$$

b)

$$\begin{aligned}
\varphi(z, x+\gamma) &= e^{\zeta(z)(x+\gamma)}\ \frac{\sigma\Big(z - x - \gamma - i\frac{\beta}{2}\Big)}{\sigma\Big(x+\gamma+i\frac{\beta}{2}\Big)} \\
&= -e^{\zeta(z)(x+\gamma)}\ \frac{\sigma\Big(-z + x + \gamma + i\frac{\beta}{2}\Big)}{\sigma\Big(x+\gamma+i\frac{\beta}{2}\Big)} \\
&= -e^{\zeta(z)(x+\gamma)}\ \frac{\sigma\Big(-z + x + i\frac{\beta}{2}\Big)e^{\eta_1(-z+x+i\frac{\beta}{2}+\frac{\gamma}{2})}}{\sigma\Big(x+i\frac{\beta}{2}\Big)e^{\eta_1(x+i\frac{\beta}{2}+\frac{\gamma}{2})}} \\
&= e^{\zeta(z)(x+\gamma)}e^{-\eta_1 z}\ \frac{\sigma\Big(z - x - i\frac{\beta}{2}\Big)}{\sigma\Big(x+i\frac{\beta}{2}\Big)} \\
&= e^{\zeta(z)\gamma - \eta_1 z}\ \varphi(z, x)
\end{aligned}$$

c)

$$\begin{aligned}
\xi(z+\gamma) &= e^{\gamma\zeta(z+\gamma)-(z+\gamma)\eta_1} &&= e^{\gamma\zeta(z)-z\eta_1} = \xi(z) \\
\xi(z+i\beta) &= e^{\gamma\zeta(z+i\beta)-(z+i\beta)\eta_1} &&= e^{\gamma\eta_2 - i\beta\eta_1}e^{\gamma\zeta(z)-z\eta_1} \\
& &&= \xi(z)
\end{aligned}$$

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Set $\Gamma = \gamma\mathbb{Z}$ and

$$V(x) = 2\wp(x + i\tfrac{\beta}{2}) \in C^\infty_{\mathbb{R}}(\mathbb{R}/\Gamma)$$
$$H = \big(i\tfrac{d}{dx}\big)^2 + V(x)$$

The Lamé equation is

$$-\tfrac{d^2}{dx^2}\phi + 2\wp(x + i\tfrac{\beta}{2})\phi = \lambda\phi \qquad \text{(S.8)}$$

A solution $\phi(k,x)$ of (S.8) that satisfies

$$\phi(k, x+\gamma) \ = \ e^{i\gamma k}\phi(k,x) \qquad \text{(S.9)}$$

is called a Bloch solution with energy λ and quasimomentum k.

Lemma S.11 says that, for each $z \in \mathbb{C} \setminus \big(\gamma\mathbb{Z} \oplus i\beta\mathbb{Z}\big)$, $\varphi(z,x)$ is a Bloch solution of the Lamé equation with energy $\lambda = \lambda(z)$ and quasimomentum $k = k(z)$.

The energy λ and multiplier $\xi = e^{\gamma ik}$ are fully parameterized by

$$\lambda(z) \; = \; -\wp(z) \qquad\qquad \xi(z) \; = \; e^{\gamma\zeta(z)-z\eta_1}$$

That is, the boundary value problem (S.8), (S.9) has a nontrivial solution if and only if $(\lambda, e^{i\gamma k}) = (\lambda(z), \xi(z))$, for some $z \in \mathbb{C} \setminus \big(\gamma\mathbb{Z} \oplus i\beta\mathbb{Z}\big)$.

Idea of Proof. Unless $2z \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$, the functions $\varphi(z,x)$ and $\varphi(-z,x)$ are linearly independent (Lemma S.12) solutions of (S.8) for $\lambda(z) = \lambda(-z)$. As a second order ordinary differential equation, (S.8) only has two linearly independent solutions for each fixed value of λ.

For $z \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$, $\lambda(z)$ is not finite.

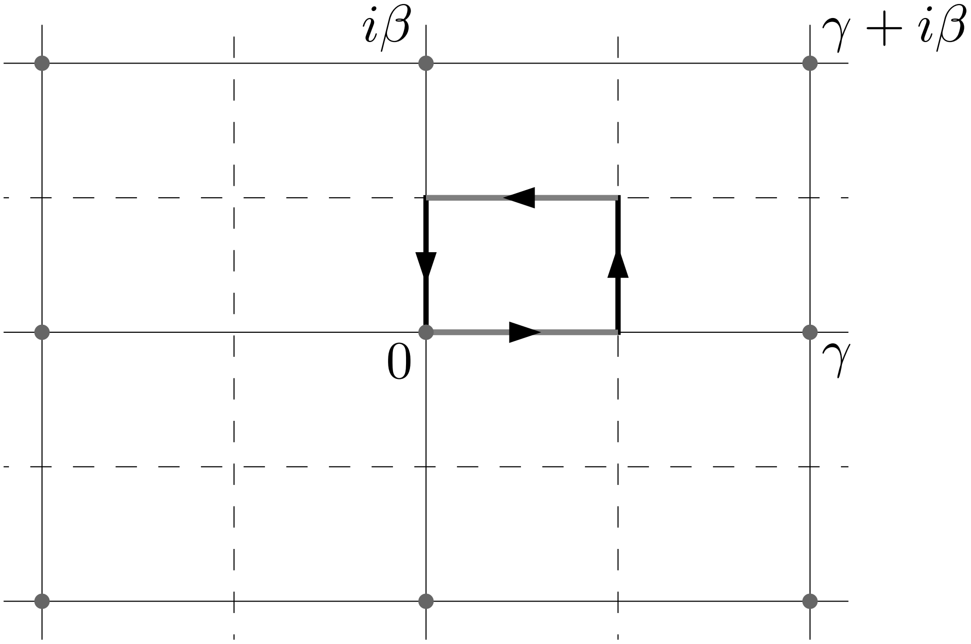
For $2z \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$ with $z \notin \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$, $\lambda'(z) = 0$ and the second linearly independent solution is $\frac{\partial}{\partial z}\varphi(z,x)$.

Theorem S.13 *Set*

$$\Lambda_1 = -\wp(\tfrac{\gamma}{2}) \qquad \Lambda_2 = -\wp(\tfrac{\gamma}{2} + i\tfrac{\beta}{2}) \qquad \Lambda_3 = -\wp(i\tfrac{\beta}{2})$$

Then $\Lambda_1, \Lambda_2, \Lambda_3$ *are real,* $\Lambda_1 < \Lambda_2 < \Lambda_3$ *and the spectrum of* $H = \big(i\frac{d}{dx}\big)^2 + 2\wp(x + i\frac{\beta}{2})$ *is* $[\Lambda_1, \Lambda_2] \cup [\Lambda_3, \infty)$.

Proof: If, for given values of λ and k, the boundary value problem (S.8), (S.9) has a nontrivial solution and **if** k **is real** then λ is in the spectrum of H. We know that all such λ's are also real.



Imagine walking along the path in the z–plane that follows the four line segments from 0 to $\frac{\gamma}{2}$ to $\frac{\gamma}{2} + i\frac{\beta}{2}$ to

$i\frac{\beta}{2}$ and back to 0. As $\overline{\wp(z)} = \wp(\bar{z})$, $\wp(-z) = \wp(z)$ and $\wp(z-\gamma) = \wp(z-i\beta) = \wp(z)$, $\lambda(z) = -\wp(z)$ remains real throughout the entire excursion. Near $z = 0$,

$$\lambda(z) = -\wp(z) \approx -\tfrac{1}{z^2}$$

so λ starts out near $-\infty$ at the beginning of the walk and moves continuously to $+\infty$ at the end of the walk. Furthermore, as

$$\begin{aligned} \wp(z) = \wp(z') \ \textit{if and only if}\ & z - z' \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z} \\ \textit{or}\ & z + z' \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}. \end{aligned}$$

λ never takes the same value twice on the walk, because no two distinct points $z,\ z'$ on the walk obey $z \pm z' \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$.

- On the first quarter of the walk, from $z = 0$ to $z = \frac{\gamma}{2}$, $\lambda(z)$ increases from $-\infty$ to $\Lambda_1 = -\wp(\frac{\gamma}{2})$. But we cannot put these λ's into the spectrum of H because $k(z)$ is pure imaginary on this part of the walk.

- On the second quarter of the walk, from $z = \frac{\gamma}{2}$ to $z = \frac{\gamma}{2} + i\frac{\beta}{2}$, $\lambda(z)$ increases from Λ_1 to $\Lambda_2 = -\wp(\frac{\gamma}{2} + i\frac{\beta}{2})$. As $k(z)$ is pure real on this part of the walk, so these λ's are in the spectrum of H.
- On the third quarter of the walk, from $z = \frac{\gamma}{2} + i\frac{\beta}{2}$ to $z = i\frac{\beta}{2}$, $\lambda(z)$ increases from Λ_2 to $\Lambda_3 = -\wp(i\frac{\beta}{2})$. These λ's do not go into the spectrum of H, because $k(z)$ has nonzero imaginary part.
- On the last quarter of the walk, from $z = i\frac{\beta}{2}$ back to zero, $\lambda(z)$ increases from Λ_3 to $+\infty$. These λ's are in the spectrum of H, because $k(z)$ is pure real.