The Weierstrass Function

Fix \( \beta, \gamma \in (0, \infty) \). Then

\[
\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \gamma \mathbb{Z} \oplus i \beta \mathbb{Z}, \omega \neq 0} \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} : \mathbb{C} \to \mathbb{C}
\]

is the Weierstrass function with primitive periods \( \gamma, i\beta \).

It obeys

a) \( \wp(z) \) is analytic on \( \mathbb{C} \setminus (\gamma \mathbb{Z} \oplus i\beta \mathbb{Z}) \).

b) \( \wp(z + \zeta) = \wp(z) \) for all \( \zeta \in \gamma \mathbb{Z} \oplus i\beta \mathbb{Z} \).

c) \( \wp(-z) = \wp(z) \) and \( \overline{\wp(z)} = \wp(\bar{z}) \)

\[\Rightarrow \wp(x + in\frac{\beta}{2}), \wp(iy + n\frac{\gamma}{2}) \text{ are real } \forall x, y \in \mathbb{R}, n \in \mathbb{Z}\]

d) Let \( c \in \mathbb{C} \). Then \( \wp(z) = c \) for exactly two \( z \)'s in each fundamental domain.

\[\Rightarrow \wp(z) = \wp(z') \text{ if and only if } z - z' \in \gamma \mathbb{Z} \oplus i\beta \mathbb{Z} \text{ or } z + z' \in \gamma \mathbb{Z} \oplus i\beta \mathbb{Z} \]

If \( z \notin \gamma \mathbb{Z} \oplus i\beta \mathbb{Z} \) but \( 2z \in \gamma \mathbb{Z} \oplus i\beta \mathbb{Z} \), \( \wp'(z) = 0 \).
Theorem W.2 Let \( f(z) \) be a nonconstant meromorphic function that is periodic with respect to \( \Omega = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z} \). Suppose that \( f(z) \) has poles of order \( n_1, \cdots, n_k \) at \( p_1 + \Omega, \cdots, p_k + \Omega \) and is analytic elsewhere. Let \( c \) be any complex number. Suppose that \( f(z) - c \) has zeroes of order \( m_1, \cdots, m_h \) at \( z_1 + \Omega, \cdots, z_h + \Omega \) and is nonzero elsewhere. Then

\[
\sum_{i=1}^{h} m_i = \sum_{i=1}^{k} n_k
\]

Idea of Proof. For any nonconstant meromorphic function \( f(z) \) and any domain \( D \)

\[
\int_{\partial D} \frac{f'(z)}{f(z)} \, dz = 2\pi i [\# \text{ zeroes in } D - \# \text{ poles in } D]
\]

Choose \( D \) of the form

\[
\begin{array}{c}
\quad z_0 + \omega_2 \\
\quad z_0 \\
\quad z_0 + \omega_1
\end{array}
\]

with no zeroes or poles on \( D \).
Weierstrass Function Relatives

Define

\[ \zeta(z) = \frac{1}{z} + \sum_{\omega \in \gamma \mathbb{Z} \oplus i \beta \mathbb{Z}} \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \]

\[ \sigma(z) = z \prod_{\omega \in \gamma \mathbb{Z} \oplus i \beta \mathbb{Z}, \omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2} \frac{z^2}{\omega^2}} \]

They obey

a) \( \zeta(z) \) is analytic on \( \mathbb{C} \setminus (\gamma \mathbb{Z} \oplus i \beta \mathbb{Z}) \). \( \sigma(z) \) is entire and vanishes if and only if \( z \in \gamma \mathbb{Z} \oplus i \beta \mathbb{Z} \).

b) \( \zeta(z) = \frac{\sigma'(z)}{\sigma(z)} \) and \( \varphi(z) = -\zeta'(z) \).

c) There are constants \( \eta_1 \in \mathbb{R} \), \( \eta_2 \in i\mathbb{R} \) satisfying \( \eta_1 i \beta - \eta_2 \gamma = 2\pi i \) such that

\[ \zeta(z + \gamma) = \zeta(z) + \eta_1 \quad \sigma(z + i \beta) = -\sigma(z) e^{\eta_2(z + i \frac{\beta}{2})} \]
\[ \zeta(z + i \beta) = \zeta(z) + \eta_2 \quad \sigma(z + \gamma) = -\sigma(z) e^{\eta_1(z + \frac{\gamma}{2})} \]

d) \( \zeta(-z) = -\zeta(z) \) and \( \overline{\zeta(z)} = \zeta(\bar{z}) \).

\[ \Rightarrow \zeta(x) \in \mathbb{R} \ \forall x \in \mathbb{R} \text{ and } \zeta(iy) \in i\mathbb{R} \ \forall y \in \mathbb{R} \]
\[ \sigma(-z) = -\sigma(z) \text{ and } \overline{\sigma(z)} = \sigma(\bar{z}) \]
e) $\varphi(u + v) + \varphi(u) + \varphi(v) = \left[\zeta(u + v) - \zeta(u) - \zeta(v)\right]^2$

**Idea of Proof.** For each fixed $v \in \mathbb{C} \setminus (\gamma \mathbb{Z} \oplus i\beta \mathbb{Z})$, both the left and right hand sides are periodic and have double poles, with the same singular part, at each $u \in \gamma \mathbb{Z} \oplus i\beta \mathbb{Z}$ and each $u \in -v + \gamma \mathbb{Z} \oplus i\beta \mathbb{Z}$.

Define

$$k(z) = -i\left(\zeta(z) - z \frac{\eta_1}{\gamma}\right)$$

It obeys

a) $k(z)$ is analytic on $\mathbb{C} \setminus (\gamma \mathbb{Z} \oplus i\beta \mathbb{Z})$.

b) $k(z + \gamma) = k(z)$ and $k(z + i\beta) = k(z) - \frac{2\pi}{\gamma}$.

c) $k(-z) = -k(z)$ and $\overline{k(z)} = -k(\overline{z})$.

$\Rightarrow k(iy), \; k\left(iy + \frac{\gamma}{2}\right) \in \mathbb{R}$ for all $y \in \mathbb{R}$.

$k(x) \in i\mathbb{R}, \; k\left(x + i\frac{\beta}{2}\right) \in \frac{\pi}{\gamma} + i\mathbb{R}$ for all $x \in \mathbb{R}$. 
Set, for $z \in \mathbb{C} \setminus (\gamma \mathbb{Z} \oplus i\beta \mathbb{Z})$,

$$\varphi(z, x) = e^{\zeta(z)x} \frac{\sigma(z - x - i\frac{\beta}{2})}{\sigma(x + i\frac{\beta}{2})}$$

$$\lambda(z) = -\varphi(z)$$

$$k(z) = -i(\zeta(z) - z\frac{\eta_1}{\gamma})$$

$$\xi(z) = e^{\gamma i k(z)} = e^{\gamma \zeta(z) - z\eta_1}$$

**Lemma S.11**

a) $$-\frac{d^2}{dx^2} \varphi(z, x) + 2\varphi(x + i\frac{\beta}{2}) \varphi(z, x) = \lambda(z) \varphi(z, x)$$

b) $$\varphi(z, x + \gamma) = \xi(z) \varphi(z, x)$$

c) $$\xi(z + \gamma) = \xi(z) \quad \xi(z + i\beta) = \xi(z)$$
Proof: a) First observe that

\[
\frac{d}{dx} \frac{\sigma(z - x - i\frac{\beta}{2})}{\sigma(x + i\frac{\beta}{2})} = - \left[ \frac{\sigma'(z - x - i\frac{\beta}{2})}{\sigma(z - x - i\frac{\beta}{2})} + \frac{\sigma'(x + i\frac{\beta}{2})}{\sigma(x + i\frac{\beta}{2})} \right] \frac{\sigma(z - x - i\frac{\beta}{2})}{\sigma(x + i\frac{\beta}{2})}
\]

\[
= - \left[ \zeta(z - x - i\frac{\beta}{2}) + \zeta(x + i\frac{\beta}{2}) \right] \frac{\sigma(z - x - i\frac{\beta}{2})}{\sigma(x + i\frac{\beta}{2})}
\]

\[
\Rightarrow \frac{d}{dx} \varphi(z, x) = \left( \zeta(z) - \zeta(z - x - i\frac{\beta}{2}) - \zeta(x + i\frac{\beta}{2}) \right) \varphi(z, x)
\]

As \[ [\zeta(u + v) - \zeta(u) - \zeta(v)]^2 = \varphi(u + v) + \varphi(u) + \varphi(v) \]

\[
\frac{d^2}{dx^2} \varphi(z, x)
\]

\[
= \left( \zeta'(z - x - i\frac{\beta}{2}) - \zeta'(x + i\frac{\beta}{2}) \right) \varphi(z, x)
\]

\[
+ \left[ \zeta(z) - \zeta(z - x - i\frac{\beta}{2}) - \zeta(x + i\frac{\beta}{2}) \right]^2 \varphi(z, x)
\]

\[
= - \left( \varphi(z - x - i\frac{\beta}{2}) - \varphi(x + i\frac{\beta}{2}) \right) \varphi(z, x)
\]

\[
+ \left[ \zeta(z) - \zeta(z - x - i\frac{\beta}{2}) - \zeta(x + i\frac{\beta}{2}) \right]^2 \varphi(z, x)
\]

\[
= - \left( \varphi(z - x - i\frac{\beta}{2}) - \varphi(x + i\frac{\beta}{2}) \right) \varphi(z, x)
\]

\[
+ \left( \varphi(z) + \varphi(z - x - i\frac{\beta}{2}) + \varphi(x + i\frac{\beta}{2}) \right) \varphi(z, x)
\]

\[
= \left( \varphi(z) + 2\varphi(x + i\frac{\beta}{2}) \right) \varphi(z, x)
\]
b) 

\[ \varphi(z, x + \gamma) = e^{\zeta(z)(x+\gamma)} \frac{\sigma(z - x - \gamma - i\frac{\beta}{2})}{\sigma(x + \gamma + i\frac{\beta}{2})} \]

\[ = -e^{\zeta(z)(x+\gamma)} \frac{\sigma(-z + x + \gamma + i\frac{\beta}{2})}{\sigma(x + \gamma + i\frac{\beta}{2})} \frac{e\eta_1(-z+x+i\frac{\beta}{2}+\gamma)}{e\eta_1(x+i\frac{\beta}{2}+\gamma)} \]

\[ = e^{\zeta(z)(x+\gamma)} e^{-\eta_1 z} \frac{\sigma(z - x - i\frac{\beta}{2})}{\sigma(x + i\frac{\beta}{2})} \]

\[ = e^{\zeta(z)\gamma-\eta_1 z} \varphi(z, x) \]

c) 

\[ \xi(z + \gamma) = e^{\gamma\zeta(z+\gamma)-(z+\gamma)\eta_1} = e^{\gamma\zeta(z)-z\eta_1} = \xi(z) \]

\[ \xi(z + i\beta) = e^{\gamma\zeta(z+i\beta)-(z+i\beta)\eta_1} = e^{\gamma\eta_2-i\beta\eta_1} e^{\gamma\zeta(z)-z\eta_1} \]

\[ = \xi(z) \]
Set $\Gamma = \gamma \mathbb{Z}$ and
\begin{align*}
V(x) &= 2\varphi(x + i\frac{\beta}{2}) \in C^\infty_\mathbb{R}(\mathbb{R}/\Gamma) \\
H &= (i\frac{d}{dx})^2 + V(x)
\end{align*}

The Lamé equation is
\begin{equation}
-\frac{d^2}{dx^2}\phi + 2\varphi(x + i\frac{\beta}{2})\phi = \lambda\phi \tag{S.8}
\end{equation}

A solution $\phi(k, x)$ of (S.8) that satisfies
\begin{equation}
\phi(k, x + \gamma) = e^{i\gamma k}\phi(k, x) \tag{S.9}
\end{equation}
is called a Bloch solution with energy $\lambda$ and quasimomentum $k$.

Lemma S.11 says that, for each $z \in \mathbb{C} \setminus (\gamma \mathbb{Z} \oplus i\beta \mathbb{Z})$, $\varphi(z, x)$ is a Bloch solution of the Lamé equation with energy $\lambda = \lambda(z)$ and quasimomentum $k = k(z)$.
The energy $\lambda$ and multiplier $\xi = e^{\gamma ik}$ are fully parameterized by

$$\lambda(z) = -\varphi(z) \quad \xi(z) = e^{\gamma \zeta(z) - z \eta_1}$$

That is, the boundary value problem (S.8), (S.9) has a nontrivial solution if and only if $(\lambda, e^{i\gamma k}) = (\lambda(z), \xi(z))$, for some $z \in \mathbb{C} \setminus (\gamma \mathbb{Z} \oplus i\beta \mathbb{Z})$.

**Idea of Proof.** Unless $2z \in \gamma \mathbb{Z} \oplus i\beta \mathbb{Z}$, the functions $\varphi(z, x)$ and $\varphi(-z, x)$ are linearly independent (Lemma S.12) solutions of (S.8) for $\lambda(z) = \lambda(-z)$. As a second order ordinary differential equation, (S.8) only has two linearly independent solutions for each fixed value of $\lambda$.

For $z \in \gamma \mathbb{Z} \oplus i\beta \mathbb{Z}$, $\lambda(z)$ is not finite.

For $2z \in \gamma \mathbb{Z} \oplus i\beta \mathbb{Z}$ with $z \notin \gamma \mathbb{Z} \oplus i\beta \mathbb{Z}$, $\lambda'(z) = 0$ and the second linearly independent solution is $\frac{\partial}{\partial z} \varphi(z, x)$.
Theorem S.13 Set

\[ \Lambda_1 = -\varphi \left( \frac{\gamma}{2} \right) \quad \Lambda_2 = -\varphi \left( \frac{\gamma}{2} + i\beta \right) \quad \Lambda_3 = -\varphi \left( i\frac{\beta}{2} \right) \]

Then \( \Lambda_1, \Lambda_2, \Lambda_3 \) are real, \( \Lambda_1 < \Lambda_2 < \Lambda_3 \) and the spectrum of \( H = \left( i\frac{d}{dx} \right)^2 + 2\varphi(x + i\frac{\beta}{2}) \) is \( [\Lambda_1, \Lambda_2] \cup [\Lambda_3, \infty) \).

**Proof:** If, for given values of \( \lambda \) and \( k \), the boundary value problem (S.8), (S.9) has a nontrivial solution and if \( k \) is real then \( \lambda \) is in the spectrum of \( H \). We know that all such \( \lambda \)'s are also real.

Imagine walking along the path in the \( z \)-plane that follows the four line segments from 0 to \( \frac{\gamma}{2} \) to \( \frac{\gamma}{2} + i\frac{\beta}{2} \) to
$i\frac{\beta}{2}$ and back to 0. As $\varphi(z) = \varphi(\bar{z})$, $\varphi(-z) = \varphi(z)$ and $\varphi(z - \gamma) = \varphi(z - i\beta) = \varphi(z)$, $\lambda(z) = -\varphi(z)$ remains real throughout the entire excursion. Near $z = 0$,

$$\lambda(z) = -\varphi(z) \approx -\frac{1}{z^2}$$

so $\lambda$ starts out near $-\infty$ at the beginning of the walk and moves continuously to $+\infty$ at the end of the walk. Furthermore, as

$$\varphi(z) = \varphi(z') \text{ if and only if } z - z' \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$$

or $z + z' \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$.

$\lambda$ never takes the same value twice on the walk, because no two distinct points $z, z'$ on the walk obey $z \pm z' \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$.

- On the first quarter of the walk, from $z = 0$ to $z = \frac{\gamma}{2}$, $\lambda(z)$ increases from $-\infty$ to $\Lambda_1 = -\varphi(\frac{\gamma}{2})$. But we cannot put these $\lambda$'s into the spectrum of $H$ because $k(z)$ is pure imaginary on this part of the walk.
• On the second quarter of the walk, from $z = \frac{\gamma}{2}$ to $z = \frac{\gamma}{2} + i\frac{\beta}{2}$, $\lambda(z)$ increases from $\Lambda_1$ to $\Lambda_2 = -\wp(\frac{\gamma}{2} + i\frac{\beta}{2})$. As $k(z)$ is pure real on this part of the walk, so these $\lambda$’s are in the spectrum of $H$.

• On the third quarter of the walk, from $z = \frac{\gamma}{2} + i\frac{\beta}{2}$ to $z = i\frac{\beta}{2}$, $\lambda(z)$ increases from $\Lambda_2$ to $\Lambda_3 = -\wp(i\frac{\beta}{2})$. These $\lambda$’s do not go into the spectrum of $H$, because $k(z)$ has nonzero imaginary part.

• On the last quarter of the walk, from $z = i\frac{\beta}{2}$ back to zero, $\lambda(z)$ increases from $\Lambda_3$ to $+\infty$. These $\lambda$’s are in the spectrum of $H$, because $k(z)$ is pure real.