Constructive Many-Body Theory

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Dedicated to Elliott H. Lieb on the occasion of his sixtieth birthday

Abstract We review the rôle played by renormalization, particle-number symmetry breaking, Goldstone bosons and the renormalization group in a mathematically rigorous construction of many-Fermion models at temperature zero.

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§I Introduction

This review is devoted to the problem of rigorously constructing a class of standard many body models, at temperature zero, in dimensions $d \geqslant 2$. Benfatto and Gallavotti and coworkers [BG1,2, BGM] are working on similar models, but have been concentrating on the one dimensional case.

The physical system that these models are designed to describe consists of a gas of fermions with prescribed density, possibly together with a crystal lattice of ions. The fermions interact with each other. If there is a lattice, it provides a periodic background potential. As well, the ions may oscillate and then the fermions interact with the lattice motion through the mediation of phonons.

The models are formally characterized by the generating functional

$$\mathcal{S}(\phi,\bar\phi) \;=\; \log \tfrac{1}{\mathcal{Z}} \int e^{[\bar\psi\phi+\bar\phi\psi]} e^{-\mathcal{V}(\psi,\bar\psi)}\; e^{-\int đk\left(ik_0 e(\mathbf{k},\mu_0)\right)\bar\psi_k\psi_k} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma} \tag{I.1}$$

for the connected, Euclidean Green's functions, where the interaction

$$\begin{aligned}\mathcal{V}(\psi,\bar\psi) \;&=\; \tfrac{\lambda}{2}\int \prod_{i=1}^{4} đk_i\ (2\pi)^{d+1}\delta(k_1{+}k_2{-}k_3{-}k_4)\ \bar\psi_{k_1}\psi_{k_3}\,\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle\,\bar\psi_{k_2}\psi_{k_4} \\ &=\; \tfrac{\lambda}{2}\sum_{\sigma,\tau\in\{\uparrow,\downarrow\}}\int \prod_{i=1}^{4} đk_i\ (2\pi)^{d+1}\delta(k_1{+}k_2{-}k_3{-}k_4)\,\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle\,\bar\psi_{k_1,\sigma}\bar\psi_{k_2,\tau}\psi_{k_4,\tau}\psi_{k_3,\sigma}\end{aligned} \tag{I.2}$$

and the source term

$$[\bar\psi\phi+\bar\phi\psi] = \int đk\ \left(\bar\psi_k\phi_k+\bar\phi_k\psi_k\right) = \int d\xi\ \left(\bar\psi(\xi)\phi(\xi)+\bar\phi(\xi)\psi(\xi)\right)$$

The denominator $\mathcal{Z}$ is chosen so that $\mathcal{S}(0,0) = 0$.

In these expressions, the internal and external electron fields are vectors

$$\psi(\xi)=\begin{pmatrix}\psi(\xi,\uparrow)\\ \psi(\xi,\downarrow)\end{pmatrix} \qquad \phi(\xi)=\begin{pmatrix}\phi(\xi,\uparrow)\\ \phi(\xi,\downarrow)\end{pmatrix}$$

$$\bar\psi(\xi)=\left(\,\bar\psi(\xi,\uparrow)\quad \bar\psi(\xi,\downarrow)\,\right) \qquad \bar\phi(\xi)=\left(\,\bar\phi(\xi,\uparrow)\quad \bar\phi(\xi,\downarrow)\,\right)$$

whose components $\psi(\xi,\sigma),\phi(\xi,\sigma),\bar\psi(\xi,\sigma),\bar\phi(\xi,\sigma),\ \xi=(t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^d,\ \sigma\in\{\uparrow,\downarrow\}$, are generators of an infinite dimensional Grassmann algebra over $\mathbb{C}$. That is, the fields anticommute with each other. The remaining notations used in (I.1,2) are $k = (k_0, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d$

$$\begin{aligned} \not{d}k &= \frac{dk_0}{2\pi}\, \not{d}\mathbf{k} = \frac{d^{d+1}k}{(2\pi)^{d+1}} \\ \psi_{k,\sigma} &= \int d\xi\, e^{i\langle k,\xi\rangle_-} \psi(\xi,\sigma) \qquad \text{with} \quad \langle k,\xi\rangle_- = -k_0 t + \langle \mathbf{k}, \mathbf{x}\rangle \\ e(\mathbf{k},\mu_0) &= \frac{\mathbf{k}^2}{2\mathbf{m}} - \mu_0 \end{aligned}$$

The chemical potential μ_0 controls the density of the system.

The abstract interaction $\langle k_1, k_2 | \mathrm{V} | k_3, k_4\rangle$ is assumed to be real and rotation, reflection and time reversal invariant. Precisely,

$$\begin{aligned} \langle k_1, k_2 | \mathrm{V} | k_3, k_4\rangle &= \langle Rk_1, Rk_2 | \mathrm{V} | Rk_3, Rk_4\rangle \quad \text{for all } R \in O(d) \\ \langle k_1, k_2 | \mathrm{V} | k_3, k_4\rangle &= \langle Tk_1, Tk_2 | \mathrm{V} | Tk_3, Tk_4\rangle \end{aligned} \tag{S1}$$

where $Rk = (k_0, R\mathbf{k})$, $Tk = (-k_0, \mathbf{k})$. We shall also assume

$$\big\langle k_1, k_2 \big| V \big| k_3, k_4 \big\rangle = \big\langle -k_3, k_2 \big| V \big| -k_1, k_4\big\rangle = \big\langle k_1, -k_4 \big| V \big| k_3, -k_2\big\rangle \tag{S2}$$

Example 1: If a gas of electrons interacts through a spin independent two body potential $V(|\mathbf{x}-\mathbf{y}|)$, then

$$\langle k_1, k_2 | \mathrm{V} | k_3, k_4\rangle = \widetilde{V}(\mathbf{k}_1 - \mathbf{k}_3)$$

satisfies (S1) and (S2). The Hamiltonian for this system is similar to the exponent of (I.1) with $\phi = \bar\phi = 0$ except that the momentum integrals only run over $\mathbb{R}^d$ (there are no zero components $k_0, (k_i)_0$), the ik_0 term is absent and the sign $-\lambda$ changes to $+\lambda$. In the Hamiltonian, $\bar\psi_{\mathbf{k},\sigma}$ and $\psi_{\mathbf{k},\sigma}$ are operators that create and annihilate, respectively, a particle of momentum $\mathbf{k}$ and spin σ.

Example 2: If the electron gas of Example 1 is coupled to a free jellium phonon field π by

$$\gamma \int \not{d}p\, \not{d}q \sqrt{\tfrac{\omega(\mathbf{q})}{2}}\; \theta\big(\omega_D - \omega(\mathbf{q})\big)\, \bar\psi_{p+q}\psi_p \left(\pi_q + \bar\pi_{-q}\right)$$

where $\omega(\mathbf{q})$ is the free phonon dispersion relation, and the jellium field is integrated out, then

$$\langle k_1, k_2 | \mathrm{V} | k_3, k_4\rangle = \widetilde{V}(\mathbf{k}_1 - \mathbf{k}_3) - \gamma^2 \theta\big(\omega_D - \omega(\mathbf{k}_1 - \mathbf{k}_3)\big)^2 \, \tfrac{\omega(\mathbf{k}_1-\mathbf{k}_3)^2}{(k_1-k_3)_0^2 + \omega(\mathbf{k}_1-\mathbf{k}_3)^2}$$

also satisfies (S1) and (S2). Here, θ smoothly restricts the coupling to phonons with frequency less than the Debye frequency ω_D.

Example 3: To obtain the generating functional for the Green's functions of a gas of electrons interacting with a lattice Γ of ions, (I.1) is modified by replacing plane waves $e^{i\langle k,\xi\rangle_-}$ by the Bloch waves for a periodic Schrödinger operator $-\Delta_{\mathbf{x}} + \sum_{a\in\Gamma} U(\mathbf{x}-a)$ and each integral $\int_{\mathbb{R}^{d+1}} đk$ by $\sum_\nu \int_{\mathbb{R}\times(\mathbb{R}^d/\Gamma^\#)} đk$ where $\Gamma^\#$ is the lattice dual to Γ and the sum is over the band index ν. For example, the kinetic term $\int đk\ \big(ik_0 e(\mathbf{k},\mu_0)\big)\bar\psi_k\psi_k$ becomes

$$\sum_\nu \int_{\mathbb{R}\times(\mathbb{R}^d/\Gamma^\#)} đk\ \big(ik_0(\varepsilon_\nu(\mathbf{k})-\mu_0)\big)\bar\psi_{\nu,k}\psi_{\nu,k}$$

Here, $\varepsilon_\nu(\mathbf{k})$, $\nu \geqslant 1$, are the band functions. The interaction is not of the form (I.2) because crystal momentum is only conserved modulo $\Gamma^\#$. This is not a particularly serious complication. But, we shall ignore crystal lattices and, as usual, make the jellium approximation in the present discussion.

§II Renormalization of the Fermi Surface

Before considering the phenomenon, symmetry breaking, that dominates this problem, we discuss the role of renormalization in infrared models. For the free model, that is the model with $\lambda = 0$, the connected, two point Schwinger function $S_2(\xi_1,\sigma_1,\xi_2,\sigma_2)$ is given by

$$S_2(\xi_1,\sigma_1,\xi_2,\sigma_2) \;=\; \big\langle \psi(\xi_1,\sigma_1)\bar\psi(\xi_2,\sigma_2)\big\rangle - \langle\psi(\xi_1,\sigma_1)\rangle\,\big\langle\bar\psi(\xi_2,\sigma_2)\big\rangle$$

where the free expectation

$$\begin{aligned}\big\langle f(\psi,\bar\psi)\big\rangle \;&=\; \tfrac{1}{\mathcal{Z}_0}\int f(\psi,\bar\psi)\ e^{-\int đk\big(ik_0 e(\mathbf{k},\mu_0)\big)\bar\psi_k\psi_k}\ \prod_{k,\sigma} d\psi_{k,\sigma}\,d\bar\psi_{k,\sigma}\\ &=\; \int f(\psi,\bar\psi)\ d\mu_0(\psi,\bar\psi)\end{aligned}$$

is integration against a Grassmann-Gaussian measure $d\mu_0(\psi,\bar\psi)$. Note that the μ_0 in $d\mu_0$ is not related to the bare chemical potential μ_0 in $e(\mathbf{k},\mu_0)$. The denominator $\mathcal{Z}_0$ is chosen so that $\int 1\ d\mu_0(\psi,\bar\psi) \;=\; 1$. Mathematically, $d\mu_0(\psi,\bar\psi)$ is characterized by

$$\int e^{[\bar\phi\psi+\bar\psi\phi]}\ d\mu_0(\psi,\bar\psi) \;=\; e^{<\bar\phi,C\phi>}$$

where, the inner product $\langle\bar\phi, C\phi\rangle = \int d\xi\, d\xi'\ \bar\phi(\xi)\, C(\xi,\xi')\, \phi(\xi')$ and the covariance C is given by the free two point function

$$C(\xi_1,\xi_2) = \int \psi(\xi_1)\bar\psi(\xi_2)\, d\mu_0(\psi,\bar\psi) = \delta_{\sigma_1,\sigma_2}\int_{\mathbb{R}^{d+1}} \bar{d}k\, \frac{e^{i\langle k,\xi_1-\xi_2\rangle_-}}{ik_0 - e(\mathbf{k},\mu_0)}$$

The expression

$$\mathcal{S}(\phi,\bar\phi) = \log \tfrac{1}{\mathcal{Z}} \int e^{[\bar\psi\phi+\bar\phi\psi]-\mathcal{V}(\psi,\bar\psi)}\, d\mu_0(\psi,\bar\psi)$$

is a starting point for a rigorous construction of many-Fermion systems.

The Fourier transform of the noninteracting two point function

$$S_2(k,p)\big|_{\lambda=0} = (2\pi)^{d+1}\delta(k-p)\,\delta_{\sigma_1,\sigma_2}\,\frac{1}{ik_0 - e(\mathbf{k},\mu_0)}$$

has a singularity on the Fermi surface $k_0 = 0, |\mathbf{k}| = \sqrt{2m\mu_0}$. The magnitude

$$\frac{1}{|ik_0 - e(\mathbf{k},\mu_0)|} \approx \frac{1}{\sqrt{k_0^2 + \mathrm{const}\,(|\mathbf{k}| - \sqrt{2m\mu_0})^2}}$$

behaves like the distance to the origin of $\mathbb{R}^2$. The following elementary calculation shows that the singularity is locally L^1, but not L^α for any $\alpha \geqslant 2$.

Denote by $k_F = \sqrt{2m\mu_0}$ the radius of the Fermi surface and change variables from k to $k_0, \mathbf{k}', \eta$ where

$$\mathbf{k}' = k_F\frac{\mathbf{k}}{|\mathbf{k}|} \qquad \eta = e(\mathbf{k},\mu_0) \tag{II.1a}$$

Here, $\mathbf{k}'$, the projection of $\mathbf{k}$ onto the Fermi surface, runs over the sphere $k_F S^{d-1}$ and plays the role of all the angular variables in spherical coordinates. The role of the radial variable is taken by η. Note that, for $\mathbf{k}$ near the Fermi surface $\eta = \frac{1}{2m}(|\mathbf{k}|+k_F)(|\mathbf{k}|-k_F) \approx \mathrm{const}\,(|\mathbf{k}|-k_F)$. In terms of the new variables, Lebesgue measure becomes

$$\begin{aligned} d^d\mathbf{k} &= \left(\frac{|\mathbf{k}|}{k_F}\right)^{d-1} d\mathbf{k}'\,d|\mathbf{k}| = \frac{m}{|\mathbf{k}|}\left(\frac{|\mathbf{k}|}{k_F}\right)^{d-1} d\mathbf{k}'\,d\eta \\ &= \frac{m}{k_F}\left(1+\frac{2m}{k_F^2}\eta\right)^{d/2-1} d\mathbf{k}'\,d\eta \end{aligned} \tag{II.1b}$$

where the surface measure $d\mathbf{k}'$ on $k_F S^{d-1}$ is normalized so that $\int 1\ d\mathbf{k}'$ is the surface area of $k_F S^{d-1}$. In these variables the magnitude of $ik_0 - e(\mathbf{k},\mu_0) = ik_0 - \eta$ is precisely the distance to the origin in $\mathbb{R}^2 = \{k_0,\eta\}$. Consider any domain of integration that contains a neighbourhood of $k_0 = 0, \mathbf{k} = \mathbf{q}$ for some $\mathbf{q}$ with $|\mathbf{q}| = k_F$. Such a domain necessarily contains a region of the form $R = \{k \mid \mathbf{k}' \in \Omega,\ k_0^2 + \eta^2 < \epsilon^2\}$ with Ω being some neighbourhood of $\mathbf{q}$ on the Fermi surface and $\epsilon > 0$. For such a domain of integration, the integral

$$\begin{aligned}
\int_R d^{d+1}k \frac{1}{|ik_0 - e(\mathbf{k}, \mu_0)|^\alpha} &= \int_R dk_0 d\mathbf{k}' d\eta\ \frac{m}{k_F}\left(1 + \frac{2m}{k_F^2}\eta\right)^{d/2-1} \frac{1}{(k_0^2+\eta^2)^{\alpha/2}} \\
&\geqslant \mathrm{vol}(\Omega)\frac{m}{k_F}\left(1 - \frac{2m}{k_F^2}\epsilon\right)^{d/2-1} \int_{|k_0|^2+|\eta|^2 \leqslant \epsilon^2} dk_0 d\eta \frac{1}{(k_0^2+\eta^2)^{\alpha/2}} \\
&= \mathrm{vol}(\Omega)\frac{m}{k_F}\left(1 - \frac{2m}{k_F^2}\epsilon\right)^{d/2-1} \int_0^\epsilon dr \int_0^{2\pi} d\theta\ r\frac{1}{r^\alpha} \\
&= 2\pi\mathrm{vol}(\Omega)\frac{m}{k_F}\left(1 - \frac{2m}{k_F^2}\epsilon\right)^{d/2-1} \int_0^\epsilon dr\ \frac{1}{r^{\alpha-1}}
\end{aligned}$$

converges if and only if $\alpha < 2$.

When the interaction is turned on so that $\lambda \neq 0$, the two point function becomes $\langle \psi_k \bar{\psi}_p \rangle = (2\pi)^{d+1}\delta(k-p)S(\mu_0, \lambda, k)$ with

$$S(\mu_0, \lambda, k) = \frac{1}{ik_0 - e(\mathbf{k}, \mu_0) - \Sigma(\mu_0, \lambda, k)}$$

where the proper self energy $\Sigma(\mu_0, \lambda, k)$ obeys $\Sigma(\mu_0, 0, k) = 0$ and is given, in perturbation theory, by the sum of all one particle irreducible two legged Feynman diagrams. Expanding in powers of λ,

$$\frac{1}{ik_0 - e(\mathbf{k}, \mu_0) - \Sigma(\mu_0, \lambda, k)} = \frac{1}{ik_0 - e(\mathbf{k}, \mu_0)} - \lambda\frac{1}{ik_0 - e(\mathbf{k}, \mu_0)}\Sigma_\lambda(\mu_0, 0, k)\frac{1}{ik_0 - e(\mathbf{k}, \mu_0)} + \cdots$$

we find that the higher order terms have non-L^1 singularities on $k_0 = 0, \mathbf{k} = \sqrt{2m\mu_0}$ unless

$$\Sigma(\mu_0, \lambda, k)\big|_{k_0=0\ \ |\mathbf{k}|=\sqrt{2m\mu_0}} = 0$$

as a formal power series in λ. These non-L^1 sigularities produce divergent Feynman diagrams.

They arise because we are attempting to expand, in powers of λ a function, $S(\mu_0, \lambda, k)$, whose singular locus depends on λ. To avoid these singularities, it suffices to parametrize the models by λ and μ, with $\sqrt{2m\mu}$ being the radius k_F of the singular locus of S rather than by λ and μ_0. That is, to perform all derivatives $\frac{\partial}{\partial\lambda}$ with μ rather than μ_0 held fixed. Thus, it is necessary to distinguish between the radius k_F of the Fermi surface, defined by $e(|\mathbf{k}| = k_F) - \Sigma(\mu_0, \lambda, k_0 = 0, |\mathbf{k}| = k_F) = 0$, and $\sqrt{2m\mu_0}$ with the chemical potential μ_0 being the coefficient of the $\int d\!\!{}^-\!k \bar{\psi}_k \psi_k$ term in the action. The two need only agree in the free model. The relationship between μ and $\mu_0 = \mu + \delta\mu(\mu, \lambda)$ is determined by

$$S^{-1}(\mu + \delta\mu(\mu,\lambda), \lambda, k)\big|_{k_0=0\ \ |\mathbf{k}|=\sqrt{2m\mu}} = 0$$

It is easy to formalize the preceding discussion by introducing the renormalized generating functional

$$\begin{aligned} \mathcal{S}_R(\phi, \bar{\phi}) &= \log \tfrac{1}{\mathcal{Z}} \int e^{[\bar{\psi}\phi + \bar{\phi}\psi] - \mathcal{V}(\psi,\bar{\psi})} e^{\ -\ \delta\mu(\lambda,\mu) \int d\!\!{}^-\!k\, \bar{\psi}_k \psi_k}\, d\mu(\psi, \bar{\psi}) \\ &= \log \tfrac{1}{\mathcal{Z}} \int e^{[\bar{\psi}\phi + \bar{\phi}\psi] - \mathcal{V}_R(\psi,\bar{\psi})}\, d\mu(\psi, \bar{\psi}) \end{aligned} \tag{II.2}$$

The Grassmann-Gaussian measure has covariance $[ik_0 - e(\mathbf{k}, \mu)]^{-1}$. The counterterm $\delta\mu(\lambda, \mu)$ is the formal power series in λ uniquely determined by the renormalization condition

$$\frac{\partial^n}{\partial\lambda^n} \Sigma\Big(0, |\mathbf{k}| = k_F, \mu, \lambda\Big)\Big|_{\lambda=0} = 0\ , \quad n \geqslant 0$$

where now $k_F = \sqrt{2\mathbf{m}\mu}$. In other words, Σ vanishes on $k_0 = 0$, $|\mathbf{k}| = k_F$ and, by definition, k_F is the radius of the interacting Fermi surface. Observe that

$$e^{-\delta\mu(\lambda,\mu) \int d\!\!{}^-\!p\, \bar{\psi}_p \psi_p} d\mu(\psi, \bar{\psi}) = \tfrac{1}{\mathcal{Z}_0}\, e^{-\int d\!\!{}^-\!k \left(ik_0 e(\mathbf{k}, \mu+\delta\mu)\right) \bar{\psi}_k \psi_k} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar{\psi}_{k,\sigma}$$

Thus, (II.1) and (II.2) agree, provided $\mu_0 = \mu + \delta\mu(\mu, \lambda)$ and the counterterm can be interpreted as implementing the shift in the radius of the Fermi surface induced by $\langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle$. A similar, but substantially more complicated, procedure can be implemented for non-spherical Fermi surfaces.

By definition, the renormalized Green's functions $S_{R\,p}$ are the coefficients in the power series expansion of (II.2) in powers of $\phi, \bar{\phi}$. Further expanding in powers of λ

$$S_{R\,p} = \sum_{n \geqslant 0} \tfrac{\lambda^n}{n!} S_{R\,p,n}$$

the n^{th} order contribution is the sum

$$S_{R\,p,n} = \sum_G \mathrm{Val}_R(G)$$

of the values of all connected n^{th} order graphs G. Recall that, for the models of Example 1,

$$\mathrm{Val}(G) \;=\; \mathrm{sgn}(G) \int \prod_{i=1}^{n} dt_j\, d\mathbf{x}_j\, dt'_j\, d\mathbf{x}'_j\, V\big(|\mathbf{x}_j - \mathbf{x}'_j|\big)\, \delta(t_j - t'_j) \prod_{\ell} C\Big((t_\ell, \mathbf{x}_\ell), \overline{(t_\ell, \mathbf{x}_\ell)}\Big)$$

In the last expression, $(t_j, \mathbf{x}_j)$, $(t'_j, \mathbf{x}'_j)$ lie at the ends of the j^{th} interaction line, the product is over all particle lines ℓ of G and $(t_\ell, \mathbf{x}_\ell)$, $\overline{(t_\ell, \mathbf{x}_\ell)}$ are the positions of the incoming and outgoing vertices of ℓ. The renormalized value $\mathrm{Val}_R(G)$ of a graph G is obtained by inductively replacing the value $\mathrm{Val}(T)(k)$ of each two legged subgraph T by

$$\mathrm{Val}(T)(k_0, \mathbf{k}) \;-\; \mathrm{Val}(T)(0, k_F)$$

Note that by rotation invariance, $\mathrm{Val}(T)(k_0, \mathbf{k}) = \mathrm{Val}(T)(k_0, |\mathbf{k}|)$.

Once the chemical potential is renormalized as in the last paragraph, all Feynman diagrams are finite. (More precisely, they are almost everywhere finite functions of their external momenta. It is possible, for example, for a four legged diagram to diverge when two of its external momenta add to zero.) In particular, there are no divergences that arise from the ultraviolet behaviour [FT1], in other words the behaviour at large k, of $[ik_0 - e(\mathbf{k}, \mu)]^{-1}$. Consequently, in contrast to the more familiar (ultraviolet) field theory case, the counterterm $\delta\mu(\mu, \lambda)$ *is finite*. To allow us to concentrate on the infrared part of the model, that after all controls it at low temperature, let's even put in an ultraviolet cutoff restricting momenta to run over a compact region.

The following Theorem gives detailed bounds on Feynman diagrams. For simplicity of language, we use "two legged subdiagram" in the statement of the Theorem, to refer to a connected subdiagram that has precisely two external particle lines and no external interaction lines and "nontrivial four legged subdiagram" to refer to a connected subdiagram that has precisely four external particle lines, no external interaction lines and is of order at least two.

Theorem II.1 (Boundedness of Feynman diagrams) [FT1,2] *Let $\langle k_1, k_2 | V | k_3, k_4\rangle$ and its first two derivatives be L^∞. There is a constant K and a norm $\|\cdot\|$ such that if the graph G is of order λ^n and*

(a) if G has no two legged or nontrivial four legged subdiagrams, then $\|\mathrm{Val}(G)\| \leqslant K^n \lambda^n$

(b) if G has no two legged subdiagrams, then $\|\mathrm{Val}(G)\| \leqslant K^n n! \lambda^n$

(c) if all two legged subgraphs of G are renormalized, then $\|\mathrm{Val}_R(G)\| \leqslant K^n n! \lambda^n$

Bounds (b) and (c) apply to the sum of all n^{th} order graphs and to the n^{th} order contribution to $\delta\mu(\mu, \lambda)$. There is a small but important class of graphs that saturate bounds (b) and (c).

Here are the main lessons. First, it is necessary to renormalize two legged subdiagrams to ensure that graphs be well defined. Second, four legged subgraphs can make the value of a graph very big. To tame this second effect requires the introduction of a renormalization group flow for the four point function (see Section VI).

§III Symmetry Breaking

The moral of renormalization is simply that one should perturb about a reasonably carefully chosen free model - namely that with the correct Fermi surface. Since we are anticipating some symmetry breaking, it makes sense to go even further and perturb about a free model that is close to the symmetry broken model rather than to the Gaussian model with covariance $[ik_0 - e(\mathbf{k}, \mu)]^{-1}$. We now determine what the symmetry broken two point function looks like for $|\mathbf{k}| = k_F$ and pick a Gaussian model whose covariance mimics it.

Set $e(\mathbf{k}) = e(\mathbf{k}, \mu)$. The action, that is the exponent of (I.1) with the external fields set to zero,

$$\begin{aligned}\mathcal{A}(\psi, \bar\psi) \; &= \; -\int đk \; \big(ik_0 e(\mathbf{k})\big)\bar\psi_k \psi_k \; - \; \delta\mu(\lambda, \mu) \int đk \; \bar\psi_k \psi_k \\ &\quad - \tfrac{\lambda}{2} \int \prod_{i=1}^{4} đk_i \; (2\pi)^{d+1} \delta(k_1{+}k_2{-}k_3{-}k_4) \; \bar\psi_{k_1} \psi_{k_3} \, \langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle \, \bar\psi_{k_2} \psi_{k_4}\end{aligned} \tag{III.1}$$

has six basic symmetries. Namely

(i) Particle number:

$$\mathcal{A}(e^{i\theta}\psi, e^{-i\theta}\bar\psi) = \mathcal{A}(\psi, \bar\psi) \qquad \forall \; e^{i\theta} \in U(1)$$

(ii) Spin:

$$\mathcal{A}(g\,\psi, \bar\psi\, g^{-1}) = \mathcal{A}(\psi, \bar\psi) \qquad \forall \; g \in SU(2)$$

(iii) Spatial rotations and reflections:

$$\mathcal{A}(R\psi, R\bar\psi) = \mathcal{A}(\psi, \bar\psi) \qquad \forall \; R \in O(d)$$

where $(R\psi)(\xi,\sigma) = \psi(R^{-1}\xi,\sigma)$ and $(R\bar\psi)(\xi,\sigma) = \bar\psi(R^{-1}\xi,\sigma)$.

(iv) Translations:

$$\mathcal{A}(T_\xi\psi, T_\xi\bar\psi) = \mathcal{A}(\psi,\bar\psi) \qquad \forall\ \xi \in \mathbb{R}^{d+1}$$

where $(T_\xi\psi)_{k,\sigma} = e^{i<k,\xi>_-}\psi_{k,\sigma}$ and $(T_\xi\bar\psi)_{k,\sigma} = e^{-i<k,\xi>_-}\bar\psi_{k,\sigma}$

(v) Time reversal:

$$\mathcal{A}(\psi,\bar\psi)^{\#} = \mathcal{A}(\psi,\bar\psi)$$

where # is the involution on the Grassmann algebra defined by $\psi_k^{\#} = \psi_{Tk}$, $\bar\psi_k^{\#} = \bar\psi_{Tk}$ and by complex conjugation of scalars.

(vi) Charge conjugation:

$$\mathcal{A}(i\bar\psi^t, i\psi^t) = \mathcal{A}(\psi,\bar\psi)$$

To verify (v), note that

$$\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle = \langle k_3,k_4|\mathrm{V}|k_1,k_2\rangle$$

follows from the reflection invariance and symmetry (S2) of V. Observe that, in contrast to the other symmetries, neither time reversal nor charge conjugation commute with the number symmetry. However, their product

(vii) CT:

$$\mathcal{A}(\psi,\bar\psi)^{\mathrm{CT}} = \mathcal{A}(\psi,\bar\psi)$$

does commute with the number symmetry. Here, CT is the involution on the Grassmann algebra defined by $\psi_k^{\mathrm{CT}} = i\bar\psi_{Tk}^t$, $\bar\psi_k^{\mathrm{CT}} = i\psi_{Tk}^t$ and by complex conjugation of scalars.

By definition, a general symmetry $\mathcal{U}$ of the action is broken if

$$\mathcal{A}\big(\mathcal{U}(\psi,\bar\psi)\big) = \mathcal{A}(\psi,\bar\psi) \qquad \text{but} \qquad \mathcal{S}\big(\mathcal{U}(\phi,\bar\phi)\big) \neq \mathcal{S}(\phi,\bar\phi)$$

In this section we study the situation when the number symmetry is broken

$$\mathcal{S}(e^{i\theta}\phi, e^{-i\theta}\bar\phi) \neq \mathcal{S}(\phi,\bar\phi)$$

but symmetries (ii,iii,iv) and (vii) above are inherited by the generating functional.

Recall that the functional derivative $\frac{\delta}{\delta\bar\phi(\xi_1,\sigma_1)}$ acting on the left of a monomial moves the factor $\bar\phi(\xi_1,\sigma_1)$ all the way to the left with the appropriate sign and deletes it.

Similarly, the functional derivative $\frac{\delta}{\delta\phi(\xi_2,\sigma_2)}$ acting on the right of a monomial moves the factor $\phi(\xi_2,\sigma_2)$ all the way to the right with the appropriate sign and deletes it. For example,

$$\begin{aligned}
\frac{\delta}{\delta\bar\phi(\xi_1,\sigma_1)}\, e^{[\bar\phi\psi+\bar\psi\phi]}\, \frac{\delta}{\delta\phi(\xi_2,\sigma_2)} &= \psi(\xi_1,\sigma_1)\, e^{[\bar\phi\psi+\bar\psi\phi]}\, \frac{\delta}{\delta\phi(\xi_2,\sigma_2)} \\
&= \psi(\xi_1,\sigma_1)\, e^{[\bar\phi\psi+\bar\psi\phi]}\, \bar\psi(\xi_2,\sigma_2) \\
&= \psi(\xi_1,\sigma_1)\bar\psi(\xi_2,\sigma_2)\, e^{[\bar\phi\psi+\bar\psi\phi]}
\end{aligned}$$

so that

$$\frac{\delta}{\delta\bar\phi(\xi_1,\sigma_1)}\, \mathcal{S}(\phi,\bar\phi)\, \frac{\delta}{\delta\phi(\xi_2,\sigma_2)}\Big|_{\phi=\bar\phi=0} = \big\langle \psi(\xi_1,\sigma_1)\bar\psi(\xi_2,\sigma_2)\big\rangle - \langle\psi(\xi_1,\sigma_1)\rangle\,\big\langle\bar\psi(\xi_2,\sigma_2)\big\rangle$$

where the interacting expectation value

$$\big\langle f(\psi,\bar\psi)\big\rangle = \tfrac{1}{\mathcal{Z}}\int f(\psi,\bar\psi)\; e^{-\mathcal{V}(\psi,\bar\psi)\; - \;\delta\mu(\lambda,\mu)\int đk\,\bar\psi_k\psi_k}\, d\mu(\psi,\bar\psi)$$

The symmetry

(viii):

$$\begin{pmatrix}\psi_{k,\uparrow}\\ \psi_{k,\downarrow}\end{pmatrix} \mapsto \begin{pmatrix} e^{i\phi} & 0\\ 0 & e^{-i\phi}\end{pmatrix}\begin{pmatrix}\psi_{k,\uparrow}\\ \psi_{k,\downarrow}\end{pmatrix}$$
$$(\,\bar\psi_{k,\uparrow}\quad \bar\psi_{k,\downarrow}\,) \mapsto (\,\bar\psi_{k,\uparrow}\quad \bar\psi_{k,\downarrow}\,)\begin{pmatrix} e^{-i\phi} & 0\\ 0 & e^{i\phi}\end{pmatrix}$$

is a subgroup of $SU(2)$. Applying this symmetry to the first derivatives of $\mathcal{S}$ we obtain $\Big\langle \overset{(-)}{\psi}_{k,\sigma}\Big\rangle = e^{\pm i\phi}\Big\langle\overset{(-)}{\psi}_{k,\sigma}\Big\rangle$ which forces

$$\langle\psi_{k,\sigma}\rangle = \big\langle\bar\psi_{k,\sigma}\big\rangle = 0$$

for all $k,\ \sigma$. Thus the second derivative of $\mathcal{S}$ simplifies to,

$$\frac{\delta}{\delta\bar\phi(\xi_1,\sigma_1)}\, \mathcal{S}(\phi,\bar\phi)\, \frac{\delta}{\delta\phi(\xi_2,\sigma_2)}\Big|_{\phi=\bar\phi=0} = \big\langle \psi(\xi_1,\sigma_1)\bar\psi(\xi_2,\sigma_2)\big\rangle$$

There are sixteen two point expectation values

$$\begin{matrix}
\langle\psi_{k\uparrow}\psi_{p\uparrow}\rangle & \langle\psi_{k\uparrow}\psi_{p\downarrow}\rangle & \langle\psi_{k\downarrow}\psi_{p\uparrow}\rangle & \langle\psi_{k\downarrow}\psi_{p\downarrow}\rangle\\
\langle\psi_{k\uparrow}\bar\psi_{p\uparrow}\rangle & \langle\psi_{k\uparrow}\bar\psi_{p\downarrow}\rangle & \langle\psi_{k\downarrow}\bar\psi_{p\uparrow}\rangle & \langle\psi_{k\downarrow}\bar\psi_{p\downarrow}\rangle\\
\langle\bar\psi_{k\uparrow}\psi_{p\uparrow}\rangle & \langle\bar\psi_{k\uparrow}\psi_{p\downarrow}\rangle & \langle\bar\psi_{k\downarrow}\psi_{p\uparrow}\rangle & \langle\bar\psi_{k\downarrow}\psi_{p\downarrow}\rangle\\
\langle\bar\psi_{k\uparrow}\bar\psi_{p\uparrow}\rangle & \langle\bar\psi_{k\uparrow}\bar\psi_{p\downarrow}\rangle & \langle\bar\psi_{k\downarrow}\bar\psi_{p\uparrow}\rangle & \langle\bar\psi_{k\downarrow}\bar\psi_{p\downarrow}\rangle
\end{matrix}$$

obtained by differentiating $\mathcal{S}$ twice with respect to $\overset{(-)}{\phi}(\xi,\sigma)$. By conservation of momentum, that is translation invariance, the distributions in the first and fourth rows vanish unless $k=-p$ while those in the second and third vanish unless $k=p$. By anticommutativity, the second row determines the third row and the second elements in the first and last rows determine the third elements. We next show that $\langle\psi_{k\uparrow}\bar\psi_{p\downarrow}\rangle$ and $\langle\psi_{k\downarrow}\bar\psi_{p\uparrow}\rangle$ vanish by $SU(2)$ invariance. Applying symmetry (viii)

$$\langle\psi_{k\uparrow}\bar\psi_{p\downarrow}\rangle \;=\; e^{2i\phi}\,\langle\psi_{k\uparrow}\bar\psi_{p\downarrow}\rangle \qquad\qquad \langle\psi_{k\downarrow}\bar\psi_{p\uparrow}\rangle \;=\; e^{-2i\phi}\,\langle\psi_{k\downarrow}\bar\psi_{p\uparrow}\rangle$$

forcing them to be zero. Similarly, applying symmetry (viii) to the four corners forces them to be zero.

Thus there are eight

$$\begin{pmatrix}\langle\psi_{k\uparrow}\bar\psi_{p\uparrow}\rangle & \langle\psi_{k\uparrow}\psi_{-p\downarrow}\rangle\\ \langle\bar\psi_{-k\downarrow}\bar\psi_{p\uparrow}\rangle & \langle\bar\psi_{-k\downarrow}\psi_{-p\downarrow}\rangle\end{pmatrix} \qquad\qquad \begin{pmatrix}\langle\bar\psi_{k\uparrow}\psi_{p\uparrow}\rangle & \langle\psi_{-k\downarrow}\psi_{p\uparrow}\rangle\\ \langle\bar\psi_{k\uparrow}\bar\psi_{-p\downarrow}\rangle & \langle\psi_{-k\downarrow}\bar\psi_{-p\downarrow}\rangle\end{pmatrix}$$

potentially nonzero two point expectation values with the second matrix determined by the first and all matrix elements vanishing unless $k=p$. The off diagonal entries of these matrices vanish when the number symmetry (i) is preserved, but may become nonzero when (i) is broken. Note that $\psi_{k\uparrow}\psi_{-k\downarrow}$ is a Cooper pair of momentum k.

Because of the above structure, it is algebraically convenient to combine the four internal physical fields $\psi_{k\uparrow}$, $\psi_{-k\downarrow}$, $\bar\psi_{k\uparrow}$ and $\bar\psi_{-k\downarrow}$ into a pair of 2-vectors

$$\begin{aligned}\mathbf{\Psi}(k) &= \begin{pmatrix}\mathbf{\Psi}^1(k)\\ \mathbf{\Psi}^2(k)\end{pmatrix} = \begin{pmatrix}\psi_{k\uparrow}\\ \bar\psi_{-k\downarrow}\end{pmatrix}\\ \bar{\mathbf{\Psi}}(k) &= (\,\bar{\mathbf{\Psi}}_1(k) \quad \bar{\mathbf{\Psi}}_2(k)\,) = (\,\bar\psi_{k\uparrow} \quad \psi_{-k\downarrow}\,)\end{aligned}$$

called “Nambu fields”. The external physical fields are combined into

$$\begin{aligned}\mathbf{\Phi}(k) &= \begin{pmatrix}\mathbf{\Phi}^1(k)\\ \mathbf{\Phi}^2(k)\end{pmatrix} = \sigma^3\begin{pmatrix}\phi_{k\uparrow}\\ \bar\phi_{-k\downarrow}\end{pmatrix}\\ \bar{\mathbf{\Phi}}(k) &= (\,\bar{\mathbf{\Phi}}_1(k) \quad \bar{\mathbf{\Phi}}_2(k)\,) = (\,\bar\phi_{k\uparrow} \quad \phi_{-k\downarrow}\,)\,\sigma^3\end{aligned}$$

Note that the external fields are twisted by σ^3, the third of the Pauli matrices

$$\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad \sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix} \qquad \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

The twist is introduced so that the source terms $[\bar\phi\psi+\bar\psi\phi]=[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]$.

With this vector notation all potentially nonzero expectation values are contained in

$$\Big(\big\langle \mathbf{\Psi}(k) \bar{\mathbf{\Psi}}(p) \big\rangle \Big) = \begin{pmatrix} \langle \psi_{k\uparrow} \bar\psi_{p\uparrow} \rangle & \langle \psi_{k\uparrow} \psi_{-p\downarrow} \rangle \\ \langle \bar\psi_{-k\downarrow} \bar\psi_{p\uparrow} \rangle & \langle \bar\psi_{-k\downarrow} \psi_{-p\downarrow} \rangle \end{pmatrix} \;=\; (2\pi)^{d+1} \delta(k-p) S(k)$$

and

$$\Big(\big\langle \bar{\mathbf{\Psi}}^t(k) \mathbf{\Psi}^t(p) \big\rangle \Big) \;=\; -\Big(\big\langle \mathbf{\Psi}(p) \bar{\mathbf{\Psi}}(k) \big\rangle \Big)^t \;=\; -(2\pi)^{d+1} \delta(k-p) S(k)^t$$

while

$$\Big(\big\langle \mathbf{\Psi}(k) \mathbf{\Psi}^t(p) \big\rangle \Big) \;=\; \begin{pmatrix} \langle \psi_{k\uparrow} \psi_{p\uparrow} \rangle & \langle \psi_{k\uparrow} \bar\psi_{-p\downarrow} \rangle \\ \langle \bar\psi_{-k\downarrow} \psi_{p\uparrow} \rangle & \langle \bar\psi_{-k\downarrow} \bar\psi_{-p\downarrow} \rangle \end{pmatrix}$$

and $\Big(\big\langle \bar{\mathbf{\Psi}}^t(k) \bar{\mathbf{\Psi}}(p) \big\rangle \Big)$ remain identically zero.

In terms of Nambu fields, the action (III.1) becomes

$$\begin{aligned} \mathcal{A} = -\int đk \; \bar{\mathbf{\Psi}}_k \big(ik_0 e(\mathbf{k}) \sigma^3 \big) \mathbf{\Psi}_k \;-\; \delta\mu(\lambda,\mu) \int đk \; \bar{\mathbf{\Psi}}_k \sigma^3 \mathbf{\Psi}_k \\ -\frac{\lambda}{2} \int đs\, đt\, đq \, \Big(\bar{\mathbf{\Psi}}_{t+\frac{q}{2}} \sigma^3 \mathbf{\Psi}_{s+\frac{q}{2}} \Big) \Big\langle t+\tfrac{q}{2}, -t+\tfrac{q}{2} \Big| V \Big| s+\tfrac{q}{2}, -s+\tfrac{q}{2} \Big\rangle \Big(\bar{\mathbf{\Psi}}_{-t+\frac{q}{2}} \sigma^3 \mathbf{\Psi}_{-s+\frac{q}{2}} \Big) \end{aligned}$$

Because the fields always appear in $\bar{\mathbf{\Psi}}\mathbf{\Psi}$ pairs, this new notation will allow us to retain conventional looking Feynman diagrams (with no $\rightarrow\!\!\times\!\!\leftarrow$ or $\longleftrightarrow$ lines) even after we introduce a symmetry broken propagator that has a nonzero value of $\langle \psi\psi \rangle$. The generating functional (II.2) becomes

$$\mathcal{S}(\mathbf{\Phi}, \bar{\mathbf{\Phi}}) = \log \tfrac{1}{\mathcal{Z}} \int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi} + \bar{\mathbf{\Psi}}\mathbf{\Phi}]} \; e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}}) - \delta\mu(\lambda,\mu) \int đk \, \bar{\mathbf{\Psi}}(k) \sigma^3 \mathbf{\Psi}(k)} \, d\mu(\mathbf{\Psi}, \bar{\mathbf{\Psi}})$$

where

$$\mathcal{V}(\mathbf{\Psi}, \bar{\mathbf{\Psi}}) \;=\; \tfrac{\lambda}{2} \int đs\, đt\, đq \, \big(\bar{\mathbf{\Psi}}(t+\tfrac{q}{2}) \sigma^3 \mathbf{\Psi}(s+\tfrac{q}{2}) \big) \Big\langle t+\tfrac{q}{2}, -t+\tfrac{q}{2} \Big| V \Big| s+\tfrac{q}{2}, -s+\tfrac{q}{2} \Big\rangle \big(\bar{\mathbf{\Psi}}(-t+\tfrac{q}{2}) \sigma^3 \mathbf{\Psi}(-s+\tfrac{q}{2}) \big)$$

and the formal Grassmann-Gaussian measure

$$d\mu(\mathbf{\Psi}, \bar{\mathbf{\Psi}}) \;=\; \tfrac{1}{\mathcal{Z}_0} \exp\Big\{ -\int đk \; \bar{\mathbf{\Psi}}(k) \big(ik_0 \mathbb{1} e(\mathbf{k}) \sigma^3 \big) \mathbf{\Psi}(k) \Big\} \prod_{k,i} d\mathbf{\Psi}^i(k) \, d\bar{\mathbf{\Psi}}_i(k)$$

is rigorously characterized by its characteristic functional

$$\int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi} + \bar{\mathbf{\Psi}}\mathbf{\Phi}]} \, d\mu(\mathbf{\Psi}, \bar{\mathbf{\Psi}}) \;=\; e^{<\bar{\mathbf{\Phi}}, \mathbf{C}\mathbf{\Phi}>}$$

The covariance

$$\mathbf{C}(k,p) = \Big(\big\langle \boldsymbol{\Psi}(k)\bar{\boldsymbol{\Psi}}(p)\big\rangle \Big) = (2\pi)^{d+1}\delta(k-p)C(k)$$

has

$$C(k) \;=\; \frac{1}{ik_0\mathbb{1} - e(\mathbf{k})\sigma^3} \;=\; -\frac{ik_0\mathbb{1} + e(\mathbf{k})\sigma^3}{k_0^2 + e(\mathbf{k})^2} \;=\; \begin{pmatrix} [ik_0 - e(\mathbf{k})]^{-1} & 0 \\ 0 & [ik_0 + e(\mathbf{k})]^{-1} \end{pmatrix}$$

We now evaluate the two point function $\Big(\big\langle \boldsymbol{\Psi}(k)\bar{\boldsymbol{\Psi}}(p)\big\rangle \Big) \;=\; (2\pi)^{d+1}\delta(k-p)S(k)$ on the surface $k_0 = 0,\ |\mathbf{k}| = k_F$. Define the real numbers Δ_1 and Δ_2 by

$$\frac{\Delta_1 - i\Delta_2}{\Delta_1^2 + \Delta_2^2} \;=\; -S_{12}(0, |\mathbf{k}| = k_F)$$

Using

Lemma III.1 [FMRT2] *Suppose the generating functional* $\mathcal{S}$ *inherits the symmetries (ii,iii,iv) and (vii) from the action* $\mathcal{A}$. *Then*

$$\begin{aligned} S(k_0,\mathbf{k}) &= S(k_0,|\mathbf{k}|) & \qquad S(0,\mathbf{k}) &= S(0,\mathbf{k})^* \\ S_{22}(k) &= -S_{11}(Tk) & S_{11}(k) &= \overline{S_{11}}(Tk) \\ S_{22}(k) &= \overline{S_{22}}(Tk) & S_{12}(k) &= \overline{S_{21}}(Tk) \end{aligned}$$

we conclude that

$$\frac{\Delta_1 + i\Delta_2}{\Delta_1^2 + \Delta_2^2} \;=\; -S_{21}(0, |\mathbf{k}| = k_F)$$

Let $d\mu_\Delta$ be the Grassmann-Gaussian measure with covariance

$$\mathbf{C}_\Delta \;=\; \frac{1}{ik_0\mathbb{1} - e(\mathbf{k})\sigma^3 - \Delta} \;=\; -\frac{ik_0\mathbb{1} + e(\mathbf{k})\sigma^3 + \Delta}{k_0^2 + E(\mathbf{k})^2}$$

where $\Delta \;=\; \Delta_1\sigma^1 + \Delta_2\sigma^2$ and $E(\mathbf{k})^2 \;=\; e(\mathbf{k})^2 + \Delta^2 \;=\; e(\mathbf{k})^2 + \Delta_1^2 + \Delta_2^2$. When $k_0 = 0$ and $|\mathbf{k}| = k_F$ the off-diagonal components of

$$\int \boldsymbol{\Psi}(k)\bar{\boldsymbol{\Psi}}(p)\ d\mu_\Delta(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}}) \;=\; -\frac{1}{k_0^2 + E(\mathbf{k})^2}\begin{pmatrix} ik_0 + e(\mathbf{k}) & \Delta_1 - i\Delta_2 \\ \Delta_1 + i\Delta_2 & ik_0 - e(\mathbf{k}) \end{pmatrix}(2\pi)^{d+1}\delta(k-p)$$

are exactly the off diagonal components of $S(k)$.

We want to treat the interacting Fermionic measure as a perturbation of $d\mu_\Delta$. For this reason, we multiply and divide by $e^{\int đk\ \bar{\boldsymbol\Psi}(k)\Delta\boldsymbol\Psi(k)}$ to obtain

$$\begin{aligned}
\mathcal{S}(\boldsymbol\Phi,\bar{\boldsymbol\Phi}) &= \log\tfrac{1}{\mathcal{Z}}\int e^{[\bar{\boldsymbol\Phi}\boldsymbol\Psi+\bar{\boldsymbol\Psi}\boldsymbol\Phi]}\ e^{-\mathcal{V}(\boldsymbol\Psi,\bar{\boldsymbol\Psi})-\delta\mu[\bar{\boldsymbol\Psi}\sigma^3\boldsymbol\Psi]}\,d\mu \\
&= \log\tfrac{1}{\mathcal{Z}}\int e^{[\bar{\boldsymbol\Phi}\boldsymbol\Psi+\bar{\boldsymbol\Psi}\boldsymbol\Phi]}\ e^{-\mathcal{V}(\boldsymbol\Psi,\bar{\boldsymbol\Psi})-\delta\mu[\bar{\boldsymbol\Psi}\sigma^3\boldsymbol\Psi]-[\bar{\boldsymbol\Psi}\Delta\boldsymbol\Psi]}e^{[\bar{\boldsymbol\Psi}\Delta\boldsymbol\Psi]}\,d\mu \\
&= \log\tfrac{1}{\mathcal{Z}_\Delta}\int e^{[\bar{\boldsymbol\Phi}\boldsymbol\Psi+\bar{\boldsymbol\Psi}\boldsymbol\Phi]}\ e^{-\mathcal{V}(\boldsymbol\Psi,\bar{\boldsymbol\Psi})-\delta\mu[\bar{\boldsymbol\Psi}\sigma^3\boldsymbol\Psi]-[\bar{\boldsymbol\Psi}\Delta\boldsymbol\Psi]}\,d\mu_\Delta
\end{aligned}$$

Define a new proper self-energy by

$$\mathbf{S}_2(k,p) = \frac{1}{ik_0 - e(\mathbf{k})\sigma^3 - \Delta - \boldsymbol\Sigma(k)}\ (2\pi)^{d+1}\delta(k-p)$$

Inverting,

$$\boldsymbol\Sigma(k) \ = \ S^{-1}(k) - ik_0 + e(\mathbf{k})\sigma^3 + \Delta$$

To be consistent with the definition of the physical chemical potential μ it is necessary that

$$\boldsymbol\Sigma_{11}(0,k_F) = \boldsymbol\Sigma_{22}(0,k_F) = 0$$

To be consistent with the definitions of Δ_1 and Δ_2 it is necessary that

$$S_{12}(0,k_F) \ = \ \mathbf{C}_{12}(0,k_F) \qquad\qquad S_{21}(0,k_F) \ = \ \mathbf{C}_{21}(0,k_F)$$

Since $\mathbf{C}_{11}(0,k_F) = \mathbf{C}_{22}(0,k_F) = 0$ and

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{pmatrix}^{-1} = \frac{1}{\det S}\begin{pmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11}\end{pmatrix}$$

the conditions on $\boldsymbol\Sigma_{11}(0,k_F)$, $\boldsymbol\Sigma_{11}(0,k_F)$, $S_{12}(0,k_F)$ and $S_{21}(0,k_F)$ may be combined in

$$\boldsymbol\Sigma(0,k_F) \ = \ 0 \tag{III.2}$$

We must renormalize to ensure that the condition (III.2) is fulfilled. That is, we introduce the renormalized action

$$\mathcal{A}_R(\boldsymbol\Psi,\bar{\boldsymbol\Psi}) \ = \ -\mathcal{V}(\boldsymbol\Psi,\bar{\boldsymbol\Psi}) - \int đk\ \bar{\boldsymbol\Psi}_k D(\lambda,\mu,\Delta)\boldsymbol\Psi_k - \int đk\ \bar{\boldsymbol\Psi}_k\big(ik_0 e(\mathbf{k})\sigma^3 - \Delta\big)\boldsymbol\Psi_k$$

and generating functional

$$\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}}) \;=\; \log \tfrac{1}{\mathcal{Z}_\Delta} \int e^{[\bar{\mathbf{\Phi}}\mathbf{\Psi}+\bar{\mathbf{\Psi}}\mathbf{\Phi}]}\; e^{-\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}})-\int\, đk\;\bar{\mathbf{\Psi}}(k)D\mathbf{\Psi}(k)}\, d\mu_\Delta$$

One can prove [FT2] that, for each $\Delta_1,\ \Delta_2$ and μ , the counterterm

$$D = D_1(\lambda,\mu,\Delta)\sigma^1 + D_2(\lambda,\mu,\Delta)\sigma^2 + D_3(\lambda,\mu,\Delta)\sigma^3$$

is uniquely determined as a formal power in λ by the renormalization condition (III.2). The coefficient D_3 is the difference between the bare and physical chemical potentials. On the other hand, there are no physical parameters to shift to accomodate D_1 and D_2 . Therefore, the constraints

$$D_1(\lambda,\mu,\Delta) \;=\; \Delta_1 \qquad\qquad D_2(\lambda,\mu,\Delta) \;=\; \Delta_2 \tag{III.3}$$

must be imposed to ensure that

$$\mathcal{S}_R(\mathbf{\Phi},\bar{\mathbf{\Phi}}) = \mathcal{S}(\mathbf{\Phi},\bar{\mathbf{\Phi}}) \tag{III.4}$$

In the Nambu notation the number symmetry is given by

$$\mathbf{\Psi}(k)\mapsto e^{i\theta\sigma^3}\mathbf{\Psi}(k) \qquad\qquad \bar{\mathbf{\Psi}}(k)\mapsto \bar{\mathbf{\Psi}}(k)e^{-i\theta\sigma^3}$$

The result of applying the number symmetry to a quadratic monomial is

$$\begin{aligned}\bar{\mathbf{\Psi}}(k)\sigma^j\mathbf{\Psi}(p) &\mapsto \bar{\mathbf{\Psi}}(k)e^{-i\theta\sigma^3}\sigma^j e^{i\theta\sigma^3}\mathbf{\Psi}(p)\\ &= \bar{\mathbf{\Psi}}(k)\left\{\begin{matrix} \sigma^1\cos 2\theta + \sigma^2\sin 2\theta & j=1\\ -\sigma^1\sin 2\theta+\sigma^2\cos 2\theta & j=2\\ \sigma^3 & j=3\end{matrix}\right\}\mathbf{\Psi}(p)\end{aligned}$$

so that

$$\begin{aligned}\mathcal{A}_R(e^{i\theta\sigma^3}\mathbf{\Psi},\bar{\mathbf{\Psi}}e^{-i\theta\sigma^3}) \;&=\; -\,\mathcal{V}(\mathbf{\Psi},\bar{\mathbf{\Psi}}) - \int đk\;\bar{\mathbf{\Psi}}_k R(2\theta)D(\lambda,\mu,\Delta)\mathbf{\Psi}_k\\ &\quad -\int đk\;\bar{\mathbf{\Psi}}_k\big(ik_0e(\mathbf{k})\sigma^3 - R(2\theta)\Delta\big)\mathbf{\Psi}_k\end{aligned}$$

where

$$R(\theta)(v\cdot\sigma) = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}\cdot\sigma$$

Recalling that the counterterms are uniquely determined as formal power series, we obtain

$$D(\lambda,\mu,R(2\theta)\Delta)=R(2\theta)D(\lambda,\mu,\Delta)$$

It follows from the last identity that the set of solutions of (III.3) is invariant under rotations. If (III.3) has a nonzero solution then

$$D_1^2(\lambda,\mu,\Delta)+D_2^2(\lambda,\mu,\Delta)=\Delta_1^2+\Delta_2^2 \tag{III.5}$$

determines $\Delta^2=\Delta_1^2+\Delta_2^2$ as a function of λ and μ but Δ_1/Δ_2 is completely free.

It is shown in [FT2] that, to any order of perturbation theory, (III.5) is of the form

$$\Delta=-\lambda_0\int d\!\!\!{}^-k\,\frac{\Delta}{k_0^2+e(\mathbf{k})^2+\Delta^2}\rho^2(\mathbf{k})+O(\Delta|\lambda|)+O(\Delta|\log\Delta||\lambda|^{5/4})$$

Here ρ is just the ultraviolet cutoff and

$$\lambda_0=\mathrm{Vol}\left(k_FS^{d-1}\right)^{-2}\int_{k_FS^{d-1}\times k_FS^{d-1}}dt'ds'\ \lambda\left\langle t',-t'|V|s',-s'\right\rangle$$

is the coupling constant in the zero angular momentum sector. The power $|\lambda|^{5/4}$, rather than $|\lambda|^2$ is undoubtedly just a reflection of sloppy bounds. This form will also apply to the full model, once its construction is complete. Thus, (III.5) is a nonperturbative version of the famous BCS gap equation. There is always the trivial solution $\Delta=0$. But when $\lambda_0<0$ there is another solution with

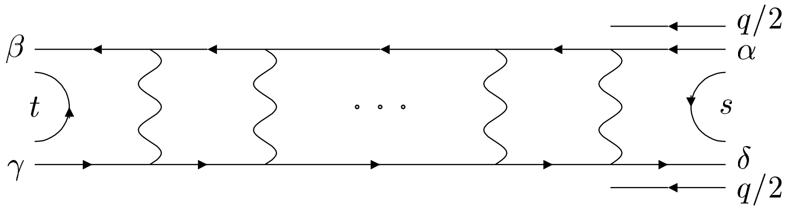
$$\Delta\approx\exp\left[\frac{-\mathrm{const}}{|\lambda_0|}\right].$$

We shall discuss Δ from another point of view in §V.

How can we tell which solution the model selects? The answer is that the trivial solution corresponds to an unstable Gaussian fixed point of the renormalization group flow that will be discussed in §VI, while the nontrivial solution corresponds to a nontrivial stable fixed point. We can see this, without having to develop all the machinery of the renormalization group, by considering a special class of graphs called ladders.

§IV Ladders

The Nambu ladder graph with external momenta s, t and q and spinor indices $\alpha, \beta, \gamma, \delta$ is

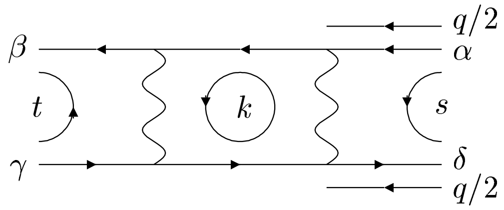


(IV.1)

An incoming arrow at a vertex represents the Nambu field $\boldsymbol{\Psi}$ and an outgoing arrow the field $\bar{\boldsymbol{\Psi}}$.

Even though the arrows on the two sides of the ladder point in opposite directions, the important components of this graph concern Cooper pairs of physical fields. The unamputated ladder is a contribution to the four point function $\left\langle \boldsymbol{\Psi}_{t+\frac{q}{2},\beta}\bar{\boldsymbol{\Psi}}_{t-\frac{q}{2},\gamma}\bar{\boldsymbol{\Psi}}_{s+\frac{q}{2},\alpha}\boldsymbol{\Psi}_{s-\frac{q}{2},\delta}\right\rangle$. Combining the $\alpha = 1$ component of the $\bar{\boldsymbol{\Psi}}_{s+\frac{q}{2},\alpha}$ with the $\delta = 2$ component of the $\boldsymbol{\Psi}_{s-\frac{q}{2},\delta}$ gives a Cooper pair $\bar{\psi}_{s+\frac{q}{2}\uparrow}\bar{\psi}_{-s+\frac{q}{2}\downarrow}$ of total momentum q and relative momentum s.

We shall retain, in the evaluation of (IV.1), only the most important part of the interaction. To get some idea as to what the most important part of the interaction is, consider the second order ladder



The value of this diagram, after amputation of the external lines, is

$$
\begin{aligned}
&-\lambda^2\int {\mathchar'26\mkern-12mu d}k\left\langle t+\tfrac{q}{2},k-\tfrac{q}{2}\middle|V\middle|k+\tfrac{q}{2},t-\tfrac{q}{2}\right\rangle\left\langle k+\tfrac{q}{2},s-\tfrac{q}{2}\middle|V\middle|s+\tfrac{q}{2},k-\tfrac{q}{2}\right\rangle(\sigma^3C(k+\tfrac{q}{2})\sigma^3)_{\beta,\alpha}(\sigma^3C(k-\tfrac{q}{2})\sigma^3)^t_{\gamma,\delta}\\
&=-\lambda^2\int {\mathchar'26\mkern-12mu d}k\left\langle \tfrac{q}{2}+t,\tfrac{q}{2}-t\middle|V\middle|\tfrac{q}{2}+k,\tfrac{q}{2}-k\right\rangle\left\langle \tfrac{q}{2}+k,\tfrac{q}{2}-k\middle|V\middle|\tfrac{q}{2}+s,\tfrac{q}{2}-s\right\rangle(\sigma^3C(k+\tfrac{q}{2})\sigma^3)_{\beta,\alpha}(\sigma^3C(k-\tfrac{q}{2})\sigma^3)^t_{\gamma,\delta}
\end{aligned}
$$

assuming V obeys the symmetry (S2). Consider the matrix element $\alpha=\beta=1, \gamma=\delta=2$. Then the propagators

$$(\sigma^3 C(k+\frac{q}{2})\sigma^3)_{\beta,\alpha} = \frac{1}{i(k_0+q_0/2)-e(\mathbf{k}+\mathbf{q}/2)}$$
$$(\sigma^3 C(k-\frac{q}{2})\sigma^3)^t_{\gamma,\delta} = \frac{1}{i(k_0-q_0/2)+e(\mathbf{k}-\mathbf{q}/2)}$$

To maximize the value of the integral we need both propagators to spend as much time simultaneously near their singularities as possible. But for $k_0+q_0/2$ and $k_0-q_0/2$ to be simultaneously zero it is neccessary that $q_0=0$ and $k_0=0$. For $e(\mathbf{k}+\mathbf{q}/2)$ and $e(\mathbf{k}-\mathbf{q}/2)$ to be simultaneously zero $\mathbf{k}$ must be on the sphere of radius k_F centered at $-\mathbf{q}/2$ and on the sphere of radius k_F centered at $\mathbf{q}/2$. The set of such $\mathbf{k}$'s is much larger when $\mathbf{q}=0$ than otherwise. The value of the matrix element is maximized when $q=0$ (in fact it diverges if and only if $q=0$) and then the dominant contributions to the integral come from k_0 near zero and $\mathbf{k}$ near the Fermi surface.

The argument of the last paragraph is made with greater precision in [FT2]. The conclusion is that the most important part of the interaction is $\langle k', -k'|V|s', -s'\rangle$ with the prime signifying that $k' = \left(0, \frac{\mathbf{k}}{|\mathbf{k}|}k_F\right)$ and s' run over the Fermi surface. Recall that k_F now means $\sqrt{2m\mu}$ rather than $\sqrt{2m\mu_0}$.

View $\langle t', -t'|V|s', -s'\rangle$ as the kernel of an integral operator on $L^2(k_F S^{d-1})$. By rotation invariance we can expand it

$$\lambda\,\langle t', -t'|V|s', -s'\rangle = \sum_{n\geqslant 0} \lambda_n \pi_n(t', s')$$

in spherical harmonics. So, $\pi_n(t', s')$ is the orthogonal projector onto the subspace of $L^2(k_F S^{d-1})$ of angular momentum n, that is, of homogeneous harmonic polynomials of degree n. We have referred a few times to "the" symmetry broken model. We shouldn't have. There are many types of symmetry breaking possible in the class of models under consideration with the type of symmetry breaking occurring in any specific model largely determined by the signs and the relative sizes of the λ_n's. We shall now restrict our attention to the most important case and assume that the coupling constant in the zero angular momentum sector, λ_0, is attractive, that is negative, and dominates the other coupling constants.

So take an interaction $\lambda\,\langle k_1, k_2|V|k_3, k_4\rangle = \lambda$ with $\lambda < 0$. This interaction lives purely in the angular momentum zero sector and is attractive. Then the value of a ladder (after amputation of its external lines) with n loops is

$$\Lambda_n(t,s,q) = -\lambda\Lambda(q)^n\sigma^3\otimes\sigma^3$$

where

$$\begin{aligned}\Lambda(q) &= -\lambda\int d\!\!{}^{-}\!k\ \left[\sigma^3 C(k+\tfrac{q}{2})\right]\otimes\left[\sigma^3 C(k-\tfrac{q}{2})^t\right]\\ &= -\lambda\int d\!\!{}^{-}\!k\ \left[\sigma^3 C(k+q)\right]\otimes\left[\sigma^3 C(k)^t\right]\end{aligned}$$

Think of q as a fixed parameter and $\Lambda(q)$ as a matrix mapping $\mathbb{C}^2\otimes\mathbb{C}^2$ to itself. So, Λ_n is independent of s and t but has two sets of double indices $\Lambda_n(t,s,q)_{(\beta,\gamma)(\alpha,\delta)}$ with α and δ being the spinor indices of the upper and lower, respectively, external legs on the right hand side of the ladder and β and γ being the spinor indices of the external legs on the left hand side of the ladder. The $n^{\rm th}$ power above refers to the $n^{\rm th}$ power of the matrix $\Lambda(q)$.

Let us first evaluate this ladder using the propagator

$$C(k) = C(k)^t = \rho(\mathbf{k})\frac{1}{ik_0 - e(\mathbf{k})\sigma^3} \ = \ -i\rho(\mathbf{k})\begin{bmatrix} k_0+ie(\mathbf{k}) & 0\\ 0 & k_0-ie(\mathbf{k})\end{bmatrix}^{-1},$$

appropriate for perturbations about the $\Delta=0$ trivial fixed point. Recall that ρ is a smooth ultraviolet cutoff and that $e(\mathbf{k}) = \frac{1}{2\mathbf{m}}\mathbf{k}^2-\mu$. Since

$$(-\lambda)(-i)(-i)\int\frac{dk_0}{2\pi}\frac{1}{k_0-ia}\frac{1}{k_0-ib} = \lambda\frac{\mathrm{sgn}(\mathrm{Re}\,a)}{a-b}\begin{cases}1 & \mathrm{Re}\,a \text{ and } \mathrm{Re}\,b \text{ of opposite sign}\\ 0 & \mathrm{Re}\,a \text{ and } \mathrm{Re}\,b \text{ of same sign}\end{cases}$$

we find, by direct computation, that two of $\Lambda(q)$'s four eigenvectors, namely $e_1\otimes e_2$ and $e_2\otimes e_1$, have eigenvalue

$$-\lambda\int_{e(\mathbf{k}+\frac{\mathbf{q}}{2})e(\mathbf{k}-\frac{\mathbf{q}}{2})>0} d\!\!{}^{-}\!\mathbf{k}\qquad\frac{\rho(\mathbf{k}+\mathbf{q})\rho(\mathbf{k})}{\mp iq_0+e(\mathbf{k}+\mathbf{q})+e(\mathbf{k})}\mathrm{sgn}e(\mathbf{k})$$

Here, e_1 and e_2 are the standard basis for $\mathbb{C}^2$. Set $q=0$ and make the change of variables (II.1). Then, the eigenvalue is

$$\begin{aligned}-\lambda\frac{m}{(2\pi)^d k_F}&\int d\mathbf{k}'d\eta\left(1+\frac{2m}{k_F^2}\eta\right)^{d/2-1}\rho(\mathbf{k})^2\frac{1}{2|\eta|}\\ &\geqslant|\lambda|\frac{m}{(2\pi)^d k_F}\left(1-\frac{2m}{k_F^2}\epsilon\right)^{d/2-1}\int_{|\eta|\leqslant\epsilon}d\mathbf{k}'d\eta\rho(\mathbf{k})^2\frac{1}{2|\eta|}\\ &=+\infty\ .\end{aligned}$$

So $\sum_n \Lambda_n$ is a geometric series with, in this case, a ratio matrix $\Lambda(q)$ that has an eigenvalue much larger than plus one for all small q. The series diverges. The analogous calculation [FT2 (I.55)] for the renormalization group flow shows that the Gaussian fixed point at $\Delta = 0$ is unstable.

Next, recall the symmetry broken propagator

$$C(k) = C(k)^t \;=\; \rho(\mathbf{k})\frac{1}{ik_0 - e(\mathbf{k})\sigma^3 - \Delta\sigma^1} \;=\; -\rho(\mathbf{k})\frac{ik_0 + e(\mathbf{k})\sigma^3 + \Delta\sigma^1}{k_0^2 + E(\mathbf{k})^2},$$

introduced immediately following Lemma III.1. Here $E(\mathbf{k}) = \sqrt{e(\mathbf{k})^2 + \Delta^2}$. We now compute the ladder using this propagator. One rung of the symmetry broken ladder takes the value

$$\Lambda(q) = -\lambda \int đk\ \rho(\mathbf{k})\rho(\mathbf{k}+\mathbf{q}) \left[\sigma^3 \frac{i(k_0+q_0) + e(\mathbf{k}+\mathbf{q})\sigma^3 + \triangle\sigma^1}{(k_0+q_0)^2 + E(\mathbf{k}+\mathbf{q})^2}\right] \otimes \left[\sigma^3 \frac{ik_0 + e(\mathbf{k})\sigma^3 + \triangle\sigma^1}{k_0^2 + E(\mathbf{k})^2}\right]$$

In [FMRT2], the k_0 integral is evaluated explicitly and the matrix norm of $\Lambda(q)$ is bounded by

$$\|\Lambda(q)\| \leqslant |\lambda| \int đ\mathbf{k}\ \frac{\rho(\mathbf{k})^2}{|E(\mathbf{k}) + E(\mathbf{k}+\mathbf{q}) + iq_0|}$$

Define

$$\gamma = -\lambda \int đ\mathbf{k}\ \frac{\rho(\mathbf{k})^2}{2E(\mathbf{k})}$$

The strict inequality in

$$\|\Lambda(0)\| \leqslant \gamma$$

$$\|\Lambda(q)\| < \gamma \qquad \text{for } q \neq 0$$

is obvious for $q_0 \neq 0$ but is also true for $\mathbf{q} \neq 0$, because the integrand is large only when both $|\mathbf{k}| \approx k_F$ and $|\mathbf{k}+\mathbf{q}| \approx k_F$. This region is a spherical shell for $\mathbf{q} = 0$ and shrinks as $|\mathbf{q}|$ increases. In particular, it is shown, by Taylor expansion, in [FMRT2] that, for q small,

$$\|\Lambda(q)\| \leqslant \gamma - c_0|\lambda|\frac{q_0^2}{\Delta^2} - c_1|\lambda|\frac{|\mathbf{q}|^2}{\Delta^2}$$

for some nonzero constants. This then implies that the full ladder obeys

$$\left|\sum_{n=1}^{\infty} \Lambda_n(t,s,q)\right| \leqslant \frac{\Delta^2}{\Delta^2(1-\gamma)/|\lambda| + c_0 q_0^2 + c_1|\mathbf{q}|^2}$$

For $\gamma < 1$ the full ladder behaves like a massive propagator.

The BCS equation, however, tells us that, to first order in λ, $\gamma = 1$. That is, the ladder with first order rungs is massless when the BCS equation is solved to first order. But we can also view the (amputated) four point function as the sum of generalized ladders whose "rungs" consist of all channel two particle irreducible four point functions. The "rung" $\Lambda(q)$ is just the first order contribution to the generalized rung. The BCS equation should be interpreted as putting the above bound exactly on the radius of convergence of the geometric series when $q = 0$ so that

$$\left|\sum_{n=1}^{\infty} \Lambda_n(t,s,q)\right| \leqslant \frac{\Delta^2}{c_0 q_0^2 + c_1 |\mathbf{q}|^2} \tag{IV.2}$$

Considering the number of bounds used in the derivation of (IV.2), one might think that the series also converges at $q = 0$. This is not the case. A calculation in [FMRT2] shows that $\Lambda(q=0)$ has an eigenvector with eigenvalue precisely γ. The appearance of 1 as an eigenvalue is no accident. It is a consequence of a Ward identity associated with the breaking of the particle number symmetry in the BCS ground state. In fact the similarity of (IV.2) to the propagator of a massless particle is again no accident. Breaking a continuous symmetry produces a mass zero particle - the Goldstone boson. We remark that the Anderson-Higgs mechanism, which generates a mass for the Goldstone boson, is not expected to be active for short range interactions.

§V The Goldstone Boson

We have already seen, in the computations of §IV, that the dominant part of the interaction $\langle t + \frac{q}{2}, -t + \frac{q}{2} | V | s + \frac{q}{2}, -s + \frac{q}{2} \rangle$ is that with $t \approx t'$, $s \approx s'$ and $q \approx 0$. So, to see how the breaking of number symmetry leads to a Goldstone boson we consider the interaction

$$\tfrac{\lambda}{2} \int \; đs \; đt \; đq \;\; \bar\psi_{(t+\frac{q}{2})}\, \psi_{(s+\frac{q}{2})} \; \langle t', -t' | V | s', -s' \rangle \; \bar\psi_{(-t+\frac{q}{2})}\, \psi_{(-s+\frac{q}{2})}$$

Indeed, we further assume that the dominant term of the decomposition $\lambda \langle t', -t' | V | s', -s' \rangle = \sum_{n \geqslant 0} \lambda_n \pi_n(t', s')$ into spherical harmonics is the zero angular momentum contribution λ_0. That is, setting $\lambda_0 = -2g^2$, we consider the effective interaction

$$\begin{aligned} \mathcal{V}_{\rm eff} &= -2g^2 \int \; đs \; đt \; đq \;\; \bar\psi_{\uparrow(t+\frac{q}{2})}\, \bar\psi_{\downarrow(-t+\frac{q}{2})}\, \psi_{\downarrow(-s+\frac{q}{2})}\, \psi_{\uparrow(s+\frac{q}{2})} \\ &= -2g^2 \int \; đp \, đq \; \left(\int \; đt \, \bar\psi_{\uparrow(t+\frac{p}{2})}\, \bar\psi_{\downarrow(-t+\frac{p}{2})} \right) B(p,-q) \left(\int \; đs \, \psi_{\downarrow(-s+\frac{q}{2})}\, \psi_{\uparrow(s+\frac{q}{2})} \right) \end{aligned}$$

with $B(p,q) \;=\; {\scriptstyle (2\pi)^{d+1}}\delta(p+q)$. Note that, by antisymmetry,

$$\int \,đs\; đt\; đq\;\; \bar\psi_{\uparrow(t+\frac{q}{2})}\,\bar\psi_{\uparrow(-t+\frac{q}{2})}\,\psi_{\uparrow(-s+\frac{q}{2})}\,\psi_{\uparrow(s+\frac{q}{2})} = 0$$

Let (γ_1,γ_2) be a $\mathbb{C}^2$ valued Gaussian variable with the real, even covariance

$$\langle \gamma_i(p)\gamma_j(q)\rangle \;=\; \delta_{i,j}\,B(p,q)$$

Observe that the position space covariance $\langle\gamma_i(\xi_1)\gamma_j(\xi_2)\rangle \;=\; \delta_{i,j}\delta(\xi_1-\xi_2)$ is also real. Therefore, this process can be realized on $\mathcal{S}'_{\mathbb{R}}\big(\mathbb{R}^{d+1}\big)$. Set

$$\boldsymbol{\Delta}(\xi) \;=\; \gamma_1(\xi) - i\gamma_2(\xi)$$

We have

$$e^{-\mathcal{V}_{\rm eff}} \;=\; \int e^{\int đq\left(\boldsymbol{\Delta}(q)\,g\int đt\,\bar\psi_\uparrow(t+\frac{q}{2})\bar\psi_\downarrow(-t+\frac{q}{2})+\overline{\boldsymbol{\Delta}}(q)\,g\int đs\,\psi_\downarrow(-s+\frac{q}{2})\psi_\uparrow(s+\frac{q}{2})\right)}\,d\mu_{(\gamma_1,\gamma_2)}$$

since for all functions $X(q)$ and $Y(q)$

$$\begin{aligned}\int e^{\int đq\,(X(q)\boldsymbol{\Delta}(q)+Y(q)\overline{\boldsymbol{\Delta}}(q))}d\mu_{(\gamma_1,\gamma_2)} \;&=\; \int e^{\int đq\left(X(q)+Y(-q)\right)\gamma_1(q)-i\left(X(q)-Y(-q)\right)\gamma_2(q)}d\mu_{(\gamma_1,\gamma_2)}\\ &=\; e^{\frac12\int đp\,đq\left(X(p)+Y(-p)\right)B(p,q)\left(X(q)+Y(-q)\right)}e^{-\frac12\int đp\,đq\left(X(p)-Y(-p)\right)B(p,q)\left(X(q)-Y(-q)\right)}\\ &=\; e^{2\int đp\,đq\,X(p)B(p,q)Y(-q)}\end{aligned}$$

Changing variables and combining terms,

$$\begin{aligned}&\int đq\,\left(\boldsymbol{\Delta}(q)\,g\int\; đt\,\bar\psi_{\uparrow(t+\frac{q}{2})}\bar\psi_{\downarrow(-t+\frac{q}{2})} + \overline{\boldsymbol{\Delta}}(q)\,g\int\; đs\,\psi_{\downarrow(-s+\frac{q}{2})}\psi_{\uparrow(s+\frac{q}{2})}\right)\\ &=\int đq\,\left(\boldsymbol{\Delta}(q)\,g\int\; đt\,\bar\psi_{\uparrow(t+\frac{q}{2})}\bar\psi_{\downarrow(-t+\frac{q}{2})} + \overline{\boldsymbol{\Delta}}(-q)\,g\int\; đt\,\psi_{\downarrow(-t-\frac{q}{2})}\psi_{\uparrow(t-\frac{q}{2})}\right)\\ &= g\int\, đq\; đt\;\left(\,\bar\psi_{\uparrow(t+\frac{q}{2})}\quad \psi_{\downarrow(-t-\frac{q}{2})}\,\right)\begin{pmatrix}0 & \boldsymbol{\Delta}(q)\\ \overline{\boldsymbol{\Delta}}(-q) & 0\end{pmatrix}\begin{pmatrix}\psi_{\uparrow(t-\frac{q}{2})}\\ \bar\psi_{\downarrow(-t+\frac{q}{2})}\end{pmatrix}\\ &= g\int\, đq\; đt\;\bar{\boldsymbol{\Psi}}_{(t+\frac{q}{2})}\begin{pmatrix}0 & \boldsymbol{\Delta}(q)\\ \overline{\boldsymbol{\Delta}}(-q) & 0\end{pmatrix}\boldsymbol{\Psi}_{(t-\frac{q}{2})} \;=\; g\int d\xi\;\bar{\boldsymbol{\Psi}}_{(\xi)}\begin{pmatrix}0 & \boldsymbol{\Delta}(\xi)\\ \overline{\boldsymbol{\Delta}}(\xi) & 0\end{pmatrix}\boldsymbol{\Psi}_{(\xi)}\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{\Psi}(k) &= \begin{pmatrix}\boldsymbol{\Psi}^1(k)\\ \boldsymbol{\Psi}^2(k)\end{pmatrix} = \begin{pmatrix}\psi_{k\uparrow}\\ \bar\psi_{-k\downarrow}\end{pmatrix}\\ \bar{\boldsymbol{\Psi}}(k) &= \left(\,\bar{\boldsymbol{\Psi}}_1(k)\quad \bar{\boldsymbol{\Psi}}_2(k)\,\right) = \left(\,\bar\psi_{k\uparrow}\quad \psi_{-k\downarrow}\,\right)\end{aligned}$$

are the Nambu fields introduced in §III. For convenience set

$$\gamma \;=\; \gamma_1\sigma^1+\gamma_2\sigma^2 \;=\; \begin{pmatrix} 0 & \boldsymbol{\Delta}(\xi) \\ \overline{\boldsymbol{\Delta}}(\xi) & 0 \end{pmatrix}$$

Then,

$$e^{-\mathcal{V}_{\rm eff}} \;=\; \int \exp\left(g\int d\xi\;\bar{\boldsymbol{\Psi}}(\xi)\,\gamma(\xi)\,\boldsymbol{\Psi}(\xi)\right)d\mu(\gamma)$$

Performing the fermionic integration

$$\begin{aligned}\int e^{-\mathcal{V}_{\rm eff}}\,d\mu(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}}) \;&=\; \int\!\!\int \exp\left(g\int d\xi\;\bar{\boldsymbol{\Psi}}(\xi)\,\gamma(\xi)\,\boldsymbol{\Psi}(\xi)\right)d\mu(\gamma_1,\gamma_2)d\mu(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}}) \\ &=\; \int \det\big(\mathbb{1}-g\,C\gamma\big)d\mu(\gamma)\end{aligned}$$

we obtain (the exponential of) an effective interaction for the intermediate boson field γ. Here, γ is a multiplication operator in position space acting on $\mathbb{R}^{d+1}$-valued functions and C is the multiplication operator in momentum space given by

$$C(p) \;=\; -\rho(p)\,\frac{ip_0+e(\mathbf{p})\sigma^3}{p_0^2+e(\mathbf{p})^2}$$

where $\rho(p)$ is the characteristic function of the set $\left\{\, p\in\mathbb{R}^{d+1} \;\middle|\; p_0^2+e(\mathbf{p})^2<1 \,\right\}$. Thus $\rho(p)$ imposes an ultraviolet, but no infrared, cutoff on the Fermions.

The determinant $\det\big(\mathbb{1}-g\,C\gamma\big)$ is a complicated function of γ. To get some feeling for it we consider constant γ's and introduce the periodized fermionic covariance $P_j(\xi)$,

$$P_j(\xi) \;=\; \sum_c C(\xi-c)$$

The sum runs over the lattice $M^{-j}\mathbb{Z}^{d+1}$ so that P_j is periodic on a large box Λ of side M^{-j}.

Lemma V.1 *If γ is constant on Λ, then*

$$\log\det(\mathbb{1}-g\,P_j\gamma) \;=\; |\Lambda|\sum_p \tfrac{1}{|\Lambda|}\,\log\left(1+\tfrac{g^2\gamma^2\rho(p)}{p_0^2+e(\mathbf{p})^2}\right)$$

where the sum is over p in $2\pi M^j\mathbb{Z}^{d+1}$ and where with abuse of notation $\gamma^2 \;=\; (\gamma_1^2+\gamma_2^2)\mathbb{1}_2$ is identified with $(\gamma_1^2+\gamma_2^2)$.

Remark. *Notice that* $\sum_p \frac{1}{|\Lambda|} \ \to\ \int_{\mathbb{R}^{d+1}} đp$ *as* $j \to -\infty$. *On the other hand, the usual volume prefactor* $|\Lambda| = M^{-(d+1)j}$ *tends to infinity since the operator* $C\gamma$ *is not trace class when* γ *is a nonzero constant.*

Proof: Set $\gamma^\sharp \ = \ -i\sigma^3\gamma \ = \ \gamma_1\sigma^2 - \gamma_2\sigma^1$. Using this notation the momentum space kernel of the operator $C\gamma$ appearing in the determinant is

$$\begin{aligned} C(p)\gamma(p-q) &= -\rho(p)\,\frac{ip_0 + e(\mathbf{p})\sigma^3}{p_0^2 + e(\mathbf{p})^2}\,\gamma(p-q) \\ &= -i\frac{\rho(p)}{p_0^2 + e(\mathbf{p})^2}\,\big(p_0\gamma(p-q) + e(\mathbf{p})\gamma^\sharp(p-q)\big) \end{aligned}$$

Periodizing the covariance,

$$\begin{aligned} \log\det(1\!\!1 - g\,P_j\gamma) &= \mathrm{Tr}\,\log\Big(1\!\!1 - g\,\tfrac{\rho(p)}{ip_0 - e(\mathbf{p})\sigma^3}\,\gamma\Big) \\ &= -\mathrm{tr}\sum_{n\geqslant 1}\frac{1}{n}(ig)^n\sum_p\Big(\tfrac{p_0\gamma + e(\mathbf{p})\gamma^\sharp}{p_0^2+e(\mathbf{p})^2}\,\rho(p)\Big)^n \end{aligned}$$

Expanding the power $\mathrm{tr}\,\big(p_0\gamma + e(\mathbf{p})\gamma^\sharp\big)^n$ produces a sum of terms each of which contains a factor

$$\mathrm{tr}\,(\sigma^1)^k(\sigma^2)^\ell \ = \ \begin{cases} 2 & k,\ell \text{ even} \\ 0 & \text{otherwise}\end{cases}$$

with $k+\ell = n$. Consequently, the expression above is zero when n is odd. Squaring

$$\begin{aligned} \Big(\tfrac{p_0\gamma+e(\mathbf{p})\gamma^\sharp}{p_0^2+e(\mathbf{p})^2}\,\rho(p)\Big)^2 &= \Big(\tfrac{\rho(p)}{p_0^2+e(\mathbf{p})^2}\Big)^2\,\Big(p_0^2\gamma^2 + p_0e(\mathbf{p})\big(\gamma\gamma^\sharp + \gamma^\sharp\gamma\big) + e(\mathbf{p})^2(\gamma^\sharp)^2\Big) \\ &= \Big(\tfrac{\rho(p)}{p_0^2+e(\mathbf{p})^2}\Big)^2\,(p_0^2 + e(\mathbf{p})^2)\,(\gamma_1^2 + \gamma_2^2) \end{aligned}$$

Substituting the last expression

$$\begin{aligned} \mathrm{Tr}\,\log\Big(1\!\!1 - g\,\tfrac{\rho(p)}{ip_0 - e(\mathbf{p})\sigma^3}\,\gamma\Big) &= -2\sum_p\sum_{n\geqslant 1}\frac{1}{2n}(-g^2)^n\,\Big(\tfrac{\rho(p)}{p_0^2+e(\mathbf{p})^2}\,(\gamma_1^2+\gamma_2^2)\Big)^n \\ &= \sum_p \log\Big(1 + g^2\tfrac{\rho(p)}{p_0^2+e(\mathbf{p})^2}\,(\gamma_1^2+\gamma_2^2)\Big) \end{aligned}$$

■

Formally,

$$\begin{aligned}
\det(\mathbb{1} - g\,C\gamma)\,d\mu(\gamma) &= e^{\log\det(\mathbb{1}-g\,C\gamma)} e^{-\frac{1}{2}\int d\xi\,\gamma B^{-1}\gamma} \prod_{\xi\in\mathbb{R}^{d+1}} d\gamma(\xi) \\
&= e^{-\frac{1}{2}\left(\int d\xi\,\gamma(\xi)^2 - \log\det(\mathbb{1}-g\,C\gamma)\right)} \prod_{\xi\in\mathbb{R}^{d+1}} d\gamma(\xi)
\end{aligned}$$

Thus, the full effective potential in a box Λ of side M^{-j} evaluated at the constant field configuration γ is

$$M^{-(d+1)j}\left(\tfrac{1}{2}\gamma^2 - \sum_p \frac{1}{|\Lambda|}\log\left(1 + g^2\tfrac{\rho(p)}{p_0^2+e(\mathbf{p})^2}\,\gamma^2\right)\right)$$

We want to show that its graph is a Bordeaux wine bottle (also referred to as a Mexican hat) and determine its dimensions.

To do this it is convenient to replace the sum $\sum_p \frac{1}{|\Lambda|}$ by an integral and study the mean field effective potential per unit volume

$$\mathcal{E}(r) = \tfrac{1}{2}r^2 - \int đp\ \log\left(1 + g^2\tfrac{\rho(p)}{p_0^2+e(\mathbf{p})^2}\,r^2\right)$$

where $r = \sqrt{\gamma_1^2+\gamma_2^2}$. In terms of the variable $s = r^2$ (but, by abuse of notation, retaining the name $\mathcal{E}$)

$$\begin{aligned}
\mathcal{E}(s) &= \tfrac{1}{2}s - \int đp\ \log\left(1 + g^2\tfrac{\rho(p)}{p_0^2+e(\mathbf{p})^2}\,s\right) \\
\frac{d\mathcal{E}}{ds}(s) &= \tfrac{1}{2} - \int đp\ \frac{g^2\rho(p)}{p_0^2 + e(\mathbf{p})^2 + g^2 s} \\
\frac{d^2\mathcal{E}}{ds^2}(s) &= \int đp\ \frac{g^4\rho(p)}{\left(p_0^2 + e(\mathbf{p})^2 + s\right)^2}
\end{aligned}$$

Hence $\mathcal{E}(s)$ is continuous on $[0,\infty)$, is zero at $s=0$ and grows like $s/2$ at $s=\infty$. The first derivative diverges logarithmically to $-\infty$ at $s=0$ and converges to $1/2$ at $s=\infty$. The second derivative is always positive. Thus $\mathcal{E}(s)$ has a unique critical point s_* and this critical point is a global minimum.

Integrating over the angular variables, changing variables to $\eta = e(\mathbf{p})$ and then using polar coordinates to replace p_0 and η

$$\begin{aligned}
\mathcal{E}(s) &= \tfrac{1}{2}s - \tfrac{|S^{d-1}|}{(2\pi)^{d-1}} \int đp_0\, đ|\mathbf{p}|\, |\mathbf{p}|^{d-1} \log\left(1 + g^2\tfrac{\rho(p)}{p_0^2+e(\mathbf{p})^2}\,s\right) \\
&= \tfrac{1}{2}s - mk_F^{d-2}\tfrac{|S^{d-1}|}{(2\pi)^{d-1}} \int đp_0\, đ\eta\ \left(1 + \tfrac{2\mathbf{m}}{k_F^2}\eta\right)^{\frac{d-2}{2}} \log\left(1 + g^2\tfrac{\rho(p)}{p_0^2+\eta^2}\,s\right) \\
&= \tfrac{1}{2}s - mk_F^{d-2}\tfrac{|S^{d-1}|}{(2\pi)^{d}} \int_0^1 dR\ R\big(1 + O(R^2)\big)\log\left(1 + \frac{g^2 s}{R^2}\right)
\end{aligned}$$

When $d = 2$ the $O(R^2)$ term is absent. When $d > 2$ we used oddness to show that the $O(\eta)$ term vanishes.

When $d = 2$ we can explicitly evaluate the integral in

$$\begin{aligned}
\mathcal{E}(s) &= \tfrac{1}{2}s - \tfrac{m}{2\pi}\int_0^1 dR\ R\log\left(1+\frac{g^2 s}{R^2}\right) \\
&= \tfrac{1}{2}s - \tfrac{m}{4\pi}\int_0^1 dx\ \log\left(1+\frac{g^2 s}{x}\right) \\
&= \tfrac{1}{2}s - \tfrac{m}{4\pi}\Big[(x+g^2s)\log(x+g^2s) - x\log x\Big]_0^1 \\
&= \tfrac{1}{2}s - \tfrac{m}{4\pi}\Big[(1+g^2s)\log(1+g^2s) - g^2s\log(g^2s)\Big]
\end{aligned}$$

The critical point obeys

$$\frac{1}{2} - \frac{mg^2}{4\pi}\log\left(\frac{1+g^2s_*}{g^2s_*}\right) = 0$$

so that

$$s_* = \frac{1}{g^2}\frac{\exp\left\{-\frac{2\pi}{mg^2}\right\}}{1-\exp\left\{-\frac{2\pi}{mg^2}\right\}}$$

or, in terms of the intermediate boson field,

$$|\gamma|_* = \frac{1}{g}\frac{\exp\left\{-\frac{\pi}{mg^2}\right\}}{\sqrt{1-\exp\left\{-\frac{2\pi}{mg^2}\right\}}} \approx \frac{1}{g}\exp\left\{-\frac{\pi}{mg^2}\right\}$$

when the coupling constant g is small. At the critical point

$$\begin{aligned}
\mathcal{E}(s_*) &= -\frac{m}{4\pi}\log\left(1+g^2s_*\right) = \frac{m}{4\pi}\log\left(1-\exp\left\{-\frac{2\pi}{mg^2}\right\}\right) \\
&\approx -\frac{m}{4\pi}\exp\left\{-\frac{2\pi}{mg^2}\right\} \approx -\frac{m}{4\pi}(g|\gamma|_*)^2
\end{aligned}$$

and

$$\frac{d^2\mathcal{E}}{dr^2}(r_*) = 4s_*\frac{d^2\mathcal{E}}{ds^2}(s_*) = \frac{m}{\pi}\frac{g^2}{1+g^2s_*} \approx \frac{m}{\pi}g^2$$

By way of resumé, the graph of the effective potential is a Bordeaux wine bottle whose absolute minimum is at $g|\gamma|_* \approx \exp\left\{-\frac{\pi}{mg^2}\right\}$ and has depth approximately $\frac{m}{4\pi}(g|\gamma|_*)^2$ and curvature at the minimum approximately $\frac{m}{\pi}g^2$. The picture in dimensions $d > 2$ is similar. Note that the depth, $M^{-(d+1)j}g|\gamma|_*$, of the effective potential

$M^{-(d+1)j}\left(\frac{1}{2}\gamma^2 - \sum_p \frac{1}{|\Lambda|}\log\left(1+g^2\frac{\rho(p)}{p_0^2+e(\mathbf{p})^2}\gamma^2\right)\right)$ in the whole box is enormous due to the volume factor $M^{-(d+1)j}$. It is deep enough to break the symmetry of the whole model.

§VI Scales and the Renormalization Group Flow

To analyze the singularity at the Fermi surface and implement, nonperturbatively, the intuition developed in the previous sections, we divide momentum space into a 'geometric series' of shells.

Fix $M > 1$ and for each $j \in \mathbb{Z}$ set

$$1_j(r) \;=\; \begin{cases} 1 & M^{2j} \leqslant r < M^{2j+2} \\ 0 & \text{otherwise} \end{cases}$$

The partition of unity

$$1 \;=\; \sum_{j=-\infty}^{\infty} 1_j\left(k_0^2 + e(\mathbf{k})^2\right)$$

divides $\mathbb{R}^{d+1}$ into shells that force $|k_0| + \big||\mathbf{k}| - k_F\big| \approx M^j$. For simplicity, we have introduced a sharp partition of unity even though a smooth one is required for the full, technically correct analysis. Set

$$C_j(\xi_1,\xi_2) \;=\; \delta_{\sigma_1,\sigma_2}\int_{\mathbb{R}^{d+1}} đk\, \frac{e^{i\langle k,\xi_1-\xi_2\rangle_-}}{ik_0 - e(\mathbf{k})}\, 1_j\left(k_0^2+e(\mathbf{k})^2\right)$$

Summing,

$$C(\xi_1,\xi_2) \;=\; \sum_{j\leqslant 0} C_j(\xi_1,\xi_2) + \sum_{j>0} C_j(\xi_1,\xi_2)$$

To focus on the infrared end of the system we impose an ultraviolet cutoff at scale $j=0$ and for each $i \leqslant 0$ introduce internal electron fields

$$\psi_i(\xi) \;=\; \begin{pmatrix} \psi_i(\xi,\uparrow) \\ \psi_i(\xi,\downarrow) \end{pmatrix} \qquad \bar\psi_i(\xi) \;=\; \left(\, \bar\psi_i(\xi,\uparrow) \quad \bar\psi_i(\xi,\downarrow)\,\right)$$

whose components are generators of a Grassmann algebra over $\mathbb{C}$. The "infrared" generating functional

$$\mathcal{I}(\phi,\bar\phi) \;=\; \log\frac{1}{\mathcal{Z}}\int e^{-\mathcal{V}_R(\psi+\phi,\bar\psi+\bar\phi)}\prod_{i\leqslant 0} d\mu_i(\psi_i,\bar\psi_i)$$

with the renormalized interaction

$$\mathcal{V}_R(\psi,\bar\psi) \;=\; \mathcal{V}(\psi,\bar\psi) \;+\; \delta\mu(\lambda,\mu)\int đk\, \bar\psi_k\psi_k$$

where

$$\psi \;=\; \sum_{i\leqslant 0} \psi_i \qquad\qquad \bar\psi \;=\; \sum_{i\leqslant 0} \bar\psi_i$$

is our starting point for a renormalization group analysis. The new functional $\mathcal{I}$ is better adapted for iteration in the renormalization group flow than (I.1). It generates the connected Euclidean Green's functions amputated by the free propagator.

By definition, the generating functional at scale j is

$$\mathcal{I}_j(g_j) \;=\; \log \tfrac{1}{Z_j} \int e^{-\mathcal{V}_R(f^j+g_j)} \prod_{0\geqslant i>j} d\mu_i(\psi_i,\bar\psi_i) \tag{VI.1}$$

where

$$f^j \;=\; \Big(\sum_{0\geqslant i>j} \psi_i \,,\; \sum_{0\geqslant i>j} \bar\psi_i \Big) \qquad g_j \;=\; \Big(\phi + \sum_{i\leqslant j} \psi_i \,,\; \bar\phi + \sum_{i\leqslant j} \bar\psi_i \Big)$$

The denominater Z_j is chosen so that $\mathcal{I}_j(0) = 0$. Here, the components of the external field g_j are elements of the Grassman algebra generated by $\phi(\xi,\sigma)$, $\bar\phi(\xi,\sigma)$ and $\psi_i(\xi,\sigma)$, $\bar\psi_i(\xi,\sigma)$, $(\xi,\sigma) \in \mathbb{R}^{d+1} \times \{\uparrow,\downarrow\}$, $i \leqslant j$. As we successively integrate over momentum shells approaching the Fermi surface, the field g_j approaches $(\phi,\bar\phi)$. At scale zero,

$$\mathcal{I}_0(g_0) \;=\; -\mathcal{V}_R\,(g_0)$$

It is not hard to show using Gram's inequality (see, [FMRT1], Lemma 1) and a cluster expansion that the generating functional $\mathcal{I}_j(g_j)$ exists. Precisely, there is a disk D_j around the origin of radius $o(1/j)$ such that for all $p \geqslant 1$, the associated Green's functions $G_{j,p}(\xi_1,,\cdots,\xi_{2p};\lambda)$ are analytic functions of λ in D_j with

$$\max_{1\leqslant m\leqslant 2p} \sup_{\xi_m} \int \prod_{n\neq m} d\xi_n \left| G_{j,p}(\xi_1,\cdots,\xi_p) \right| \;<\; \infty$$

It is no surprise that the radius of D_j tends to zero as j tends to $-\infty$, since the spontaneously broken number symmetry, discussed in §III, generates a mass $\Delta \approx e^{-\mathrm{const}/\lambda}$ in the two point function that is not analytic in λ. However, the estimate that one obtains by a straightforward application of Gram's inequality reflects a singularity much more severe than $e^{-\mathrm{const}/\lambda}$. A much more careful analysis of the Pauli exclusion principle in the presence of a Fermi surface is required to reconcile the difference. See [FMRT 1].

Exponentiating $\mathcal{I}_j(g_j)$ and writing $g_j = g_{j-1} + (\psi_j, \bar{\psi}_j)$,

$$\begin{aligned}
\mathcal{I}_{j1}(g_{j-1}) &= \log \tfrac{1}{Z_{j-1}} \int e^{-\mathcal{V}_R\left(f^{j-1}+g_{j-1}\right)} \prod_{i>j-1} d\mu_i(\psi_i, \bar{\psi}_i) \\
&= \log \int e^{\mathcal{I}_j(g_{j-1}+(\psi_j,\bar{\psi}_j))}\, d\mu_j(\psi_j, \bar{\psi}_j) \;+\; \log \tfrac{Z_j}{Z_{j-1}} \\
&= \log \int e^{\mathcal{I}_j(g_{j-1}+(\psi_j,\bar{\psi}_j))}\, d\mu_j(\psi_j, \bar{\psi}_j) \;-\; \log \int e^{\mathcal{I}_j((\psi_j,\bar{\psi}_j))}\, d\mu_{j}(\psi_j,\bar{\psi}_j)
\end{aligned}$$

To pass from the second to the third line observe that

$$0 \;=\; \mathcal{I}_{j-1}(0) \;=\; \log \int \exp\left(\mathcal{I}_j((\psi_j, \bar{\psi}_j))\right)\, d\mu_j(\psi_j, \bar{\psi}_j) \;+\; \log \tfrac{Z_j}{Z_{j-1}}$$

By definition, the map from $\mathcal{I}_j(g_j)$ to $\mathcal{I}_{j-1}(g_{j-1})$ given by

$$\mathcal{I}_{j-1}(g_{j-1}) \;=\; \log \int e^{\mathcal{I}_j(g_{j-1}+(\psi_j,\bar{\psi}_j))}\, d\mu_j(\psi_j, \bar{\psi}_j) \;-\; \log \int e^{\mathcal{I}_j((\psi_j,\bar{\psi}_j))}\, d\mu_j(\psi_j, \bar{\psi}_j)$$

is the renormalization group transformation at scale j.

For any Grassman valued function $U(\,\cdot\,)$ on the Grassman algebra set

$$\mathcal{E}_j\left(U, g_{j-1}\right) \;=\; \log \int \exp\left(U(g_{j-1} + (\psi_j, \bar{\psi}_j))\right)\, d\mu_j(\psi_j, \bar{\psi}_j) \;-\; U(g_{j-1})$$

Then, iterating the renormalization group transform generates a solution to the first order difference equation

$$\mathcal{I}_{j-1}(g_{j-1}) \;=\; \mathcal{I}_j(g_{j-1}) \,+\, \mathcal{E}_j\left(\mathcal{I}_j, g_{j-1}\right) - \mathcal{E}_j\left(\mathcal{I}_j, 0\right) \tag{VI.2}$$

where,

$$\mathcal{I}_j(g_{j-1}) \;=\; \mathcal{I}_j\left(g_{j-1} + (\psi_j, \bar{\psi}_j)\right)\Big|_{(\psi_j,\bar{\psi}_j)=0}$$

We want to construct a solution to (VI.2) such that the limit

$$\mathcal{I}(\phi, \bar{\phi}) \;=\; \lim_{j\searrow-\infty} \mathcal{I}_j(\phi, \bar{\phi})$$

exists and satisfies the renormalization condition

$$\Sigma(0, |\mathbf{k}| = k_F, \mu, \lambda) \;=\; 0$$

introduced in §II. The renormalization condition is now a boundary condition at $j = -\infty$.

Let $f(\bar\psi,\psi)$ be any Grassman valued function homogeneous of degree $2p$. That is

$$f(\gamma\bar\psi,\gamma\psi) \;=\; \gamma^{2p} f(\bar\psi,\psi)$$

To implement the renormalization boundary condition that fixes the value of the two point function on the Fermi surface, define the two point localization operator $\mathcal{L}_2$ by

$$\mathcal{L}_2\, f(\bar\psi,\psi) \;=\; \begin{cases} f & p=0 \text{ and } f\in\mathbb{C} \\ T(0,|\mathbf{k}|=k_F)\int đk\bar\psi_k\psi_k & p=1 \text{ and } f=\int đk\,T(k)\,\bar\psi_k\psi_k \\ 0 & \text{otherwise} \end{cases}$$

The operator $\mathcal{L}_2$ extends by linearity to all formal power series on the Grassman algebra. For example,

$$\begin{aligned}
\mathcal{L}_2\,\mathcal{I}_j(g_j) &= \mathcal{L}_2 \int đk\, G_{j,2}(k)\left(\bar\phi+\sum_{i\leqslant j}\bar\psi_i\right)\left(\phi+\sum_{i\leqslant j}\psi_i\right) \\
&= G_{j,2}(0,|\mathbf{k}|=k_F)\int đk\,\left(\bar\phi+\sum_{i\leqslant j}\bar\psi_i\right)\left(\phi+\sum_{i\leqslant j}\psi_i\right) \\
&= \Sigma_j(0,|\mathbf{k}|=k_F)\int đk\,\left(\bar\phi+\sum_{i\leqslant j}\bar\psi_i\right)\left(\phi+\sum_{i\leqslant j}\psi_i\right)
\end{aligned}$$

since

$$G_{j,2}(k) \;=\; \Sigma_j(k)\left(\mathbb{1}+\sum_{n\geqslant 1}\left(C^{(j)}(k)\Sigma_j(k)\right)^n\right)$$

and the covariance of the Guassian measure $\prod_{0\geqslant i>j} d\mu_i(\psi_i,\bar\psi_i)$ obeys

$$C^{(j)}(0,|\mathbf{k}|=k_F) \;=\; \sum_{i>j} C_i(0,|\mathbf{k}|=k_F) \;=\; 0$$

by construction. Thus,

$$\lim_{j\searrow-\infty}\mathcal{L}_2\,\mathcal{I}_j(\phi,\bar\phi) \;=\; \Sigma(0,|\mathbf{k}|=k_F)\int đk\;\bar\phi\phi$$

Summarizing the discussion of the last two paragraphs, our problem is to construct a solution to the boundary value problem

$$\mathcal{I}_{j-1}(g_{j-1})-\mathcal{I}_j(g_{j-1}) \;=\; \mathcal{E}_j\left(\mathcal{I}_j,g_{j-1}\right)-\mathcal{E}_j\left(\mathcal{I}_j,0\right)$$

$$\mathcal{I}_0(g_0) \;=\; -\mathcal{V}_R\,(g_0)$$

$$\lim_{j \searrow -\infty} \mathcal{L}_2\, \mathcal{I}_j(\phi, \bar{\phi}) \;=\; 0$$

For all $k \leqslant j-1$

$$\mathcal{I}_{j-1}(g_k) - \mathcal{I}_j(g_k) \;=\; \mathcal{E}_j\,(\mathcal{I}_j, g_k) - \mathcal{E}_j\,(\mathcal{I}_j, 0)$$

where

$$\mathcal{I}_j(g_k) \;=\; \mathcal{I}_j\big(g_k + \sum_{k<i\leqslant j} (\psi_i, \bar{\psi}_i)\big)\Big|_{\substack{(\psi_i,\bar{\psi}_i)=0 \\ k<i\leqslant j}}$$

It follows that

$$\begin{aligned}
\mathcal{I}_{-1}(g_j) - \mathcal{I}_0(g_j) \;&=\; \mathcal{E}_0\,(\mathcal{I}_0, g_j) - \mathcal{E}_0\,(\mathcal{I}_0, 0) \\
\mathcal{I}_{-2}(g_j) - \mathcal{I}_{-1}(g_j) \;&=\; \mathcal{E}_{-1}\,(\mathcal{I}_{-1}, g_j) - \mathcal{E}_{-1}\,(\mathcal{I}_{-1}, 0) \\
\vdots \qquad & \qquad \vdots \\
\mathcal{I}_j(g_j) - \mathcal{I}_{j+1}(g_j) \;&=\; \mathcal{E}_{j+1}\,(\mathcal{I}_{j+1}, g_j) - \mathcal{E}_{j+1}\,(\mathcal{I}_{j+1}, 0)
\end{aligned}$$

Summing, we obtain the "integral equation"

$$\mathcal{I}_j(g_j) \;=\; \mathcal{I}_0(g_j) \,+\, \sum_{i>j} \mathcal{E}_i\,(\mathcal{I}_i, g_j) - \mathcal{E}_i\,(\mathcal{I}_i, 0)$$

The boundary value at $j = -\infty$ is

$$\lim_{j \searrow -\infty} \mathcal{L}_2\, \mathcal{I}_j(\phi, \bar{\phi}) \;=\; \mathcal{L}_2\, \mathcal{I}_0(\phi, \bar{\phi}) \,+\, \sum_{i\leqslant 0} \mathcal{L}_2\, \mathcal{E}_i\,\big(\mathcal{I}_i, \phi, \bar{\phi}\big) - \mathcal{E}_i\,(\mathcal{I}_i, 0)$$

Thus, if we chose

$$\mathcal{I}_0(g_0) \;=\; -\mathcal{V}(g_0) \;-\; \sum_{i=-\infty}^{0} \big(\mathcal{L}_2\, \mathcal{E}_i\big(\mathcal{I}_i, g_0\big) - \mathcal{E}_i\,(\mathcal{I}_i, 0)\,\big)$$

then the solution $\mathcal{I}_j(g_j)$ of the 'initialized' integral equation

$$\mathcal{I}_j(g_j) \;=\; -\mathcal{V}(g_j) \;-\; \sum_{i=-\infty}^{0} \big(\mathcal{L}_2\, \mathcal{E}_i(\mathcal{I}_i, g_j) - \mathcal{E}_i(\mathcal{I}_i, 0)\big) \;+\; \sum_{i>j} \big(\mathcal{E}_i\,(\mathcal{I}_i, g_j) - \mathcal{E}_i(\mathcal{I}_i, 0)\big)$$

is a solution of (VI.2) satisfying

$$\lim_{j \searrow -\infty} \mathcal{L}_2\, \mathcal{I}_j(\phi, \bar{\phi}) \;=\; \lim_{j \searrow -\infty} \sum_{i\leqslant j} \mathcal{L}_2\, \mathcal{E}_i\,\big(\mathcal{I}_i, \phi, \bar{\phi}\big) - \mathcal{E}_i\,(\mathcal{I}_i, 0) \;=\; 0$$

since $\mathcal{L}_2\, \mathcal{V} \;=\; 0$. It follows that $\delta\mu(\lambda, \mu)$ is determined by

$$\delta\mu(\lambda, \mu) \int đk\, \bar{\phi}\phi \;=\; \sum_{i=-\infty}^{0} \big(\,\mathcal{L}_2\, \mathcal{E}_i\,\big(\mathcal{I}_i, \phi, \bar{\phi}\big) - \mathcal{E}_i\,(\mathcal{I}_i, 0)\,\big)$$

Finally, we rewrite the initialized equation as

$$\mathcal{I}_j(g_j) \;=\; -\mathcal{V}(g_j) \;-\; \sum_{i\leqslant j} \mathcal{L}_2\, \mathcal{E}_i(\mathcal{I}_i, g_j) \;+\; \sum_{i>j} (\mathbb{1}-\mathcal{L}_2)\mathcal{E}_i\,(\mathcal{I}_i, g_j) \;+\; \sum_{i\leqslant j} \mathcal{E}_i(\mathcal{I}_i, 0) \qquad \text{(VI.3)}$$

What have we accomplished? The infrared generating functional $\mathcal{I}(\phi, \bar{\phi})$ has been expressed as a limit of approximations $\mathcal{I}_j(g_j)$, $j \leqslant 0$, that successively probe the Fermi surface singularity in the free propagator

$$\frac{1}{ik_0 - e(\mathbf{k})}$$

These cutoff functionals satisfy "integral equation " (VI.3) in which the quadratic part is carefully split at each scale by the operators $\mathcal{L}_2$ and $\mathbb{1} - \mathcal{L}_2$ into counter and renormalized terms that implement the renormalization of the chemical potential discussed in §II.

We can construct a formal power series solution to (VI.3) by iterating it, starting with $\mathcal{I}_j(g_j) = -\mathcal{V}(g_j)$. For example, after one iteration, $\mathcal{I}_j$ is

$$-\mathcal{V}(g_j) \;-\; \sum_{i\leqslant j} \mathcal{L}_2\, \mathcal{E}_i(-\mathcal{V}, g_j) \;+\; \sum_{i>j} (\mathbb{1}-\mathcal{L}_2)\mathcal{E}_i\,(-\mathcal{V}, g_j) \;+\; \sum_{i\leqslant j} \mathcal{E}_i(-\mathcal{V}, 0)$$

Repeated iteration generates a tree expansion. See [FT1,2, Ga]. Expanding $\mathcal{E}_i\,(U, g_j)$ in a Taylor series in U and then collecting terms of the same degree in $\mathcal{V}$ generates a perturbation expansion. The lines of the graphs generated by $\mathcal{E}_i$ have propagators $\frac{1_i\left(k_0^2+e(\mathbf{k})^2\right)}{ik_0-e(\mathbf{k})}$. Roughly speaking, $\mathbb{1} - \mathcal{L}_2$ replaces the value

$$\tfrac{1_{j_1}\left(k_0^2+e(\mathbf{k})^2\right)}{ik_0-e(\mathbf{k})}\, T(k)\, \tfrac{1_{j_2}\left(k_0^2+e(\mathbf{k})^2\right)}{ik_0-e(\mathbf{k})}$$

of every two legged subgraph, whose external legs lie at scales j_1, j_2 below those of its internal lines (note that the range of summation for $\sum (\mathbb{1}-\mathcal{L}_2)\mathcal{E}_i\,(\mathcal{I}_i, g_j)$ is $i > j$), by

$$\tfrac{1_{j_1}\left(k_0^2+e(\mathbf{k})^2\right)}{ik_0-e(\mathbf{k})}\, \left(T(k) - T(0, k_F)\right)\, \tfrac{1_{j_2}\left(k_0^2+e(\mathbf{k})^2\right)}{ik_0-e(\mathbf{k})}$$

The zero of $T(k) - T(0, k_F)$ on the Fermi surface moderates the singularity of the neighbouring $\frac{1}{ik_0-e(\mathbf{k})}$.

We now turn to the more subtle problem of extracting from quartic contributions to $\mathcal{I}$ a dominant "local part" that, in the language of Theorem II.1, generate dangerously large factors of $n!$. The following definition of "local part" is motivated by the discussion of the third paragraph of §IV. Suppose $f(\bar\psi,\psi)$ is a Grassmann valued function homogeneous of degree $2p$. For each $j \leqslant 0$ set

$$\mathcal{L}_{4,j}\, f(\bar\psi,\psi) \;=\; 0 \qquad \text{when } p \neq 2$$

and, for $p = 2$,

$$\begin{aligned}&\mathcal{L}_{4,j} \int đt\, đs\, đq\, K(t,s,q)\, \bar\psi_{i_1}(t+\tfrac{q}{2},\alpha)\bar\psi_{i_2}(-t+\tfrac{q}{2},\mu)\psi_{i_4}(-s+\tfrac{q}{2},\mu)\psi_{i_3}(s+\tfrac{q}{2},\alpha)\\ &\quad= \int_{|\mathbf{q}|\leqslant M^{j^*}} đt\, đs\, đq\, K(t',s',0)\bar\psi_{i_1}(t+\tfrac{q}{2},\alpha)\bar\psi_{i_2}(-t+\tfrac{q}{2},\mu)\psi_{i_4}(-s+\tfrac{q}{2},\mu)\psi_{i_3}(s+\tfrac{q}{2},\alpha)\end{aligned}$$

where the sum over repeated spin indices has been suppressed, and

$$\begin{aligned} j^* &= \tfrac{1}{2}\left(j + \max\left\{i_1,i_2,i_3,i_4\right\}\right)\\ s' &= (s_0,\mathbf{s})' \;=\; (0,\tfrac{t}{|t|}k_F)\\ &= \text{the projection of } s \text{ onto the Fermi surface}\end{aligned}$$

The operator $\mathcal{L}_{4,j}$ extends by linearity to the whole Grassmann algebra.

Observe that $\mathbf{q}$ is restricted to a ball of radius M^{j^*} centered on the origin in $\mathbb{R}^d$. The radius shrinks to zero as $j \to -\infty$. In the approximation that $\mathbf{q}$ is set to zero in the fields $\overset{(-)}{\psi}_{i_n}$, the quartic local part becomes

$$\tfrac{1}{V}\int đt\, đs\, đq^0\, K(t',s',0)\, \bar\psi_{i_1}(t+\tfrac{q^0}{2},\alpha)\bar\psi_{i_2}(-t+\tfrac{q^0}{2},\mu)\psi_{i_4}(-s+\tfrac{q^0}{2},\mu)\psi_{i_3}(s+\tfrac{q^0}{2},\alpha)$$

with the notation

$$\begin{aligned} q^0 &= (q_0,\mathbf{0})\\ \tfrac{1}{V} &= \int_{|\mathbf{q}|\leqslant M^{j^*}} đ\mathbf{q}\end{aligned}$$

The local part of a quartic is, up to sign, a reduced interaction. In fact, it is the analog in the functional integral formalism of the interaction term in the usual BCS reduced Hamiltonian

$$\sum_{\mathbf{k}\sigma} e(\mathbf{k})a^+_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma} + \sum_{\substack{|e(\mathbf{k}_1)|<\epsilon\\ |e(\mathbf{k}_2)|<\epsilon}} \langle \mathbf{k}_1,-\mathbf{k}_1|V|\mathbf{k}_2,-\mathbf{k}_2\rangle a^+_{\mathbf{k}_1,\alpha}a^+_{-\mathbf{k}_1,\beta}a_{-\mathbf{k}_2,\beta}a_{\mathbf{k}_2,\alpha}$$

The kernel $K(t',s',0)$ of the quartic local part generates a rotation invariant integral operator on $L^2\left(k_F S^{d-1}\right)$. Decomposing into spherical harmonics

$$K(t',s',0) \;=\; \sum_{n\geqslant 0}\lambda_n\pi_n(t',s')$$

Here, $\pi_n(t',s')$ is the projection onto the spherical harmonics of degree n and λ_n , $n \geqslant 0$, is the spectrum of K . Suppose that the zero angular momentum sector of K is dominant. That is, $|\lambda_0| > |\lambda_n|$, $n \geqslant 1$. Then,

$$\frac{1}{V} \int đt\, đs\, đq^0\; 2\lambda_0 \bar\psi_{i_1}(t + \tfrac{q^0}{2}, \uparrow) \bar\psi_{i_2}(-t + \tfrac{q^0}{2}, \downarrow) \psi_{i_4}(-s + \tfrac{q^0}{2}, \downarrow) \psi_{i_3}(s + \tfrac{q^0}{2}, \uparrow)$$

is the leading contribution to the quartic local part. It is the familiar BCS interaction. See, for example, [FW], p.333.

To exploit the quartic localization operator $\mathcal{L}_{4,j}$, denote by

$$\mathcal{F}_j(g_j) \;=\; \mathcal{L}_{4,j}\, \mathcal{I}_j(g_j)$$

the quartic local part of the generating functional at scale j. We have

$$\begin{aligned}
\mathcal{F}_0(g_0) \;&=\; -\mathcal{L}_{4,0}\, \mathcal{V}(g_0) \\
&=\; -\tfrac{\lambda}{2} \sum \int_{|q|\leqslant M^{0^*}} đt\, đs\, đq \big\langle t', -t' |V| s', -s' \big\rangle \bar\psi_{i_1}(t+\tfrac{q}{2},\alpha) \bar\psi_{i_2}(-t+\tfrac{q}{2},\beta) \psi_{i_4}(-s+\tfrac{q}{2},\beta) \psi_{i_3}(s+\tfrac{q}{2},\alpha) \\
&=\; \sum \int_{|q|\leqslant M^{0^*}} đt\, đs\, đq\, F_0(s',t') \bar\psi_{i_1}(t+\tfrac{q}{2},\alpha) \bar\psi_{i_2}(-t+\tfrac{q}{2},\beta) \psi_{i_4}(-s+\tfrac{q}{2},\beta) \psi_{i_3}(s+\tfrac{q}{2},\alpha)
\end{aligned}$$

where the sum is over all $i_1, i_2, i_3, i_4 \leqslant 0$ and all repeated spin indices. Recall that, $0^* = \frac{1}{2}\max\{i_1, i_2, i_3, i_4\}$. In general

$$\mathcal{F}_j(g_j) \;=\; \sum \int_{|q|\leqslant M^{j^*}} đt\, đs\, đq\, F_j(s',t') \bar\psi_{i_1}(t+\tfrac{q}{2},\alpha) \bar\psi_{i_2}(-t+\tfrac{q}{2},\beta) \psi_{i_4}(-s+\tfrac{q}{2},\beta) \psi_{i_3}(s+\tfrac{q}{2},\alpha)$$

Rewriting (VI.3) so as to isolate the quartic local part, we obtain the coupled system of "integro-differential" equations"

$$\begin{aligned}
\mathcal{I}_j(g_j) \;&=\; \mathcal{F}_j(g_j) \;+\; \sum_{i>j} \big(\mathcal{L}_{(i)} - \mathcal{L}_{(i-1)}\big)\, \mathcal{F}_i(g_i) \;-\; \sum_{i\leqslant j} \mathcal{L}_2\, \mathcal{E}_i(\mathcal{I}_i, g_j) + \sum_{i\leqslant j} \mathcal{E}_i(\mathcal{I}_i, 0) \\
&\qquad -\, (\mathbb{1} - \mathcal{L}_{(0)}) \mathcal{V}(g_j) \;+\; \sum_{i>j} (\mathbb{1} - \mathcal{L}_{(i-1)}) \mathcal{E}_i\,(\mathcal{I}_i, g_j) \\
\mathcal{F}_{j-1}(g_{j-1}) \;&=\; \mathcal{L}_{4,j-1}\, \mathcal{F}_j(g_j) \;+\; \mathcal{L}_{4,j-1}\, \mathcal{E}_j(\mathcal{I}_j, g_j)
\end{aligned} \tag{VI.4}$$

for $\mathcal{I}_j(g_j)$, $\mathcal{F}_j(g_j)\; j \leqslant 0$. Here, for each $i \leqslant 0$,

$$\mathcal{L}_{(i)} \;=\; \mathcal{L}_2 \oplus \mathcal{L}_{4,i}$$

is the direct sum of the quadratic and quartic localization operators. Observe that $\mathcal{I}_j(g_j)$ has been decomposed in (VI.4) into a local quadratic piece

$$-\sum_{i\leqslant j}\mathcal{L}_2\,\mathcal{E}_i(\mathcal{I}_i,g_j)+\sum_{i\leqslant j}\mathcal{E}_i(\mathcal{I}_i,0)$$

an effective quartic interaction

$$\mathcal{F}_j(g_j)\;+\;\sum_{i>j}\big(\mathcal{L}_{(i)}-\mathcal{L}_{(i-1)}\big)\,\mathcal{F}_i(g_i)$$

and a renormalized piece

$$-(\mathbb{1}-\mathcal{L}_{(0)})\mathcal{V}(g_j)\;+\;\sum_{i>j}(\mathbb{1}-\mathcal{L}_{(i-1)})\mathcal{E}_i\,(\mathcal{I}_i,g_j)$$

As before, we can construct a formal power series solution to (VI.4) by iteration, starting with $\mathcal{I}_j(g_j)=-\mathcal{V}(g_j)$, $\mathcal{F}_j(g_j)=-\mathcal{L}_{4,j}\mathcal{V}(g_j)$. Expanding $\mathcal{E}_i(U,g_j)$ and collecting terms generates a perturbation expansion at scale j in which the corresponding graphs contain generalized two legged vertices, generalized four legged vertices representing the effective interactions $\mathcal{F}_i(g_i)$, $i\geqslant j$, renormalized two legged insertions, and renormalized four legged insertions $(\mathbb{1}-\mathcal{L}_{4,i-1})\mathcal{E}_i\,(\mathcal{I}_i,g_j)$, $i>j$. Again, lines introduced by integration of the Gaussian measure in $\mathcal{E}_i$ have scale i.

The system of flow equations (VI.4) has been organized so as to isolate the "singular" part of the model in relatively simple local parts. To test whether or not $\mathcal{L}$ really captures the singular part of the model, and in particular whether or not $\mathcal{L}_{4,i}$ really captures the singular part of four legged graphs that produces the dangerously large factor $n!$ in Theorem II.1(c), consider a truncated flow

$$\mathcal{I}_j(g_j)\;=-(\mathbb{1}-\mathcal{L}_{(0)})\mathcal{V}(g_j)\;+\;\sum_{i>j}(\mathbb{1}-\mathcal{L}_{(i-1)})\mathcal{E}_i\,(\mathcal{I}_i,g_j)\qquad\text{(VI.4)}_{\mathrm{T}}$$

from which the local parts have been removed by hand.

Theorem VI.1 [FT2] *Let G be a graph appearing in the expansion generated by the truncated flow (VI.4)$_{\mathrm{T}}$. Then there is a norm and a positive constant K such that*

$$\Big\|\,\mathrm{Val}\big(G\big)\,\Big\|\;\leqslant\;K^n\lambda^n$$

The lesson here is that any anomalous behaviour of perturbation theory is isolated in the local parts $\mathcal{F}_i(g_i)$. Everything else, "is irrelevant" in renormalization group language. Theorem (VI.1) is a result about individual graphs.

At least in two space dimensions, one can use the Pauli exclusion principle to prove the analogous non-perturbative result. Theorem 2 of [FMRT1] says that, roughly speaking, the series

$$\sum_{n\geqslant 0} \frac{1}{n!}\left(\frac{\lambda}{2}\right)^n \sum_{\substack{\text{graphs } G \text{ without 2} \\ \text{or 4 legged sub-graphs}}} \mathrm{Val}(G)$$

is analytic in a neighborhood of $\lambda = 0$. It should be easy to extend this to show that the series

$$\sum_{n\geqslant 0} \frac{1}{n!}\left(\frac{\lambda}{2}\right)^n \sum_{\substack{\text{graphs } G \text{ without} \\ \text{local sub-graphs}}} \mathrm{Val}_{\mathrm{ren}}(G)$$

in which all subgraphs are renormalized is also analytic in a disk around zero.

On a perturbative level, the flow of the local part is studied in [FT2]. It diverges away from the trivial fixed point whenever the initial interaction

$$\lambda \langle t', -t'|V|s', -s'\rangle \;=\; \sum_{n\geqslant 0} \lambda_n \pi_n(t', s')$$

is attractive in any angular momentum sector $\pi_n(t', s')$. If the interaction is attractive in the zero sector $\lambda_0 < 0$ and this sector is dominant, that is $|\lambda_0| > |\lambda_n|,\ n > 0$, then it is shown in [FT2] that, after truncating perturbation theory at any finite order, the local part flows toward an effective theory "near" the BCS model, at least until the flow reaches energy scales in which the Goldstone Boson becomes important. It is expected that the local part remains near the BCS model even at those energy scales. We are now attempting to prove that this is indeed the case.

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