Power Series Representations for Bosonic Effective Actions

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Abstract

We develop a power series representation and estimates for an effective action of the form

$$\ln \frac{\int e^{F(\psi,\varphi)} d\mu(\varphi)}{\int e^{f(\varphi,0)} d\mu(\varphi)}$$

Here, $F(\psi, \varphi)$ is an analytic function of the real fields $\varphi(\mathbf{x}), \psi(\mathbf{x})$ indexed by $\mathbf{x}$ in a finite set X, and $d\mu(\varphi)$ is a compactly supported product measure. Such effective actions occur in the small field region for a renormalization group analysis. The customary way to analyze them is a cluster expansion, possibly preceded by a decoupling expansion. Using methods similar to a polymer expansion, we estimate the power series of the effective action without introducing an artificial decomposition of the underlying space into boxes.

Outline

- Motivation
- The Main Theorem
- Outline of the Proof - Algebraic Part
- Norms
- Outline of the Proof - Bounds

Motivation – Renormalization Group Construction Protocol

- Express all quantities of interest as functional integrals like
$$\mathcal{G}(\Psi) = \ln \frac{\int e^{\mathcal{A}(\Psi,\Phi)}\, d\mu(\Phi)}{\int e^{\mathcal{A}(0,\Phi)}\, d\mu(\Phi)}$$
- Factor the measure $d\mu(\Phi) = \prod_{\ell=1}^{\infty} d\mu_\ell(\varphi_\ell)$, with the least important degrees of freedom having index ℓ small, to express
$$\mathcal{G}(\Psi) = \ln \frac{\int e^{\mathcal{A}(\Psi,\varphi_1,\varphi_2,\cdots)} \prod_{\ell=1}^{\infty} d\mu_\ell(\varphi_\ell)}{\int e^{\mathcal{A}(0,\varphi_1,\varphi_2,\cdots)} \prod_{\ell=1}^{\infty} d\mu_\ell(\varphi_\ell)}$$
- Do the integrals one at a time. Define the "effective action at scale n" to be
$$\mathcal{A}_n(\Psi,\varphi_{n+1},\varphi_{n+2},\cdots) = \ln \frac{\int e^{\mathcal{A}(\Psi,\varphi_1,\varphi_2,\cdots)} \prod_{\ell=1}^{n} d\mu_\ell(\varphi_\ell)}{\int e^{\mathcal{A}(0,\varphi_1,\cdots,\varphi_n,0,\cdots)} \prod_{\ell=1}^{n} d\mu_\ell(\varphi_\ell)}$$

  Then
$$\mathcal{A}_n(\psi) = \ln \frac{\int e^{\mathcal{A}_{n-1}(\psi,\varphi)}\, d\mu_n(\varphi)}{\int e^{\mathcal{A}_{n-1}(0,\varphi)}\, d\mu_n(\varphi)}$$
  where $\varphi = \varphi_n$ and $\psi = (\Psi,\varphi_{n+1},\varphi_{n+2},\cdots)$.

The Main Theorem

Let $X(=\mathrm{space})$ be a finite set. Let $d\mu_0(t)$ be a normalized measure on $\mathbb{R}$ that is supported in $|t| \le r$ for some constant r. We endow $\mathbb{R}^X$ with the ultralocal product measure

$$d\mu(\varphi) = \prod_{\mathbf{x}\in X} d\mu_0\big(\varphi(\mathbf{x})\big)$$

Theorem III.4 *Let w and W be weight systems for 1 and 2 fields, respectively, that obey*

$$W(\vec{\mathbf{x}}, \vec{\mathbf{y}}) \ge (4r)^{n(\vec{\mathbf{y}})} w(\vec{\mathbf{x}})$$

If $F(\psi,\varphi)$ obeys $\|F\|_W < \frac{1}{16}$, then there is a real analytic function $f(\psi)$ such that

$$\frac{\int e^{F(\psi,\varphi)}\, d\mu(\varphi)}{\int e^{F(0,\varphi)}\, d\mu(\varphi)} = e^{f(\psi)} \qquad \text{(III.1)}$$

and

$$\|f\|_w \le \tfrac{\|F\|_W}{1-16\|F\|_W}$$

Notation

$$\begin{aligned} \mathbf{x} \in X &= \text{space, a finite set} \\ \vec{\mathbf{x}} \in \mathcal{X} &= \text{multispace} = \bigcup_{n\ge 0} X^n \\ &= \left\{\ (\mathbf{x}_1,\cdots,\mathbf{x}_n) \in X^n \ \middle|\ n \ge 0\ \right\} \end{aligned}$$

For $\vec{\mathbf{x}} = (\mathbf{x}_1,\cdots,\mathbf{x}_n) \in X^n$, $\vec{\mathbf{y}} = (\mathbf{y}_1,\cdots,\mathbf{y}_m) \in X^m$ and $\varphi : X \to \mathbb{R}$,

$$\begin{aligned} n(\vec{\mathbf{x}}) &= n \\ \vec{\mathbf{x}} \circ \vec{\mathbf{y}} &= \big(\mathbf{x}_1,\cdots,\mathbf{x}_n,\mathbf{y}_1,\cdots,\mathbf{y}_m\big) \in X^{n+m} \\ \varphi(\vec{\mathbf{x}}) &= \varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2)\cdots\varphi(\mathbf{x}_n) \\ \operatorname{supp}(\vec{\mathbf{x}}) &= \{\mathbf{x}_1,\cdots,\mathbf{x}_n\} \subset X \end{aligned}$$

If F, f are real analytic on a neighbourhood of the origin, then there are unique expansions

$$\begin{aligned} F(\psi,\varphi) &= \sum_{\vec{\mathbf{x}},\vec{\mathbf{y}}\in\mathcal{X}} A(\vec{\mathbf{x}},\vec{\mathbf{y}})\ \psi(\vec{\mathbf{x}})\varphi(\vec{\mathbf{y}}) \\ f(\psi) &= \sum_{\vec{\mathbf{x}}\in\mathcal{X}} a(\vec{\mathbf{x}})\ \psi(\vec{\mathbf{x}}) \end{aligned}$$

with $A(\vec{\mathbf{x}},\vec{\mathbf{y}})$, $a(\vec{\mathbf{x}})$ invariant under permutations of the components of $\vec{\mathbf{x}}$ and under permutations of the components of $\vec{\mathbf{y}}$.

Outline of the Proof – Algebra

Write $F(\psi,\varphi) = \sum_{\vec{\mathbf{x}},\vec{\mathbf{y}}\in\mathcal{X}} A(\vec{\mathbf{x}},\vec{\mathbf{y}})\ \psi(\vec{\mathbf{x}})\varphi(\vec{\mathbf{y}})$.

- ▷ Set

$$\alpha(\vec{\mathbf{y}}) = \sum_{\vec{\mathbf{x}}\in\mathcal{X}} A(\vec{\mathbf{x}},\vec{\mathbf{y}})\ \psi(\vec{\mathbf{x}})$$

With this notation

$$F(\psi,\varphi) = \sum_{\vec{\mathbf{y}}\in\mathcal{X}} \alpha(\vec{\mathbf{y}})\ \varphi(\vec{\mathbf{y}})$$

By factoring $e^{F(\psi,0)}$ out of the integral in the numerator of (III.1), we may assume that $F(\psi,0)=0$. Expanding the exponential

$$\begin{aligned} e^{F(\psi,\varphi)} &= \sum_{\ell=0}^{\infty} \tfrac{1}{\ell!} F(\psi,\varphi)^{\ell} \\ &= 1 + \sum_{\ell=1}^{\infty} \tfrac{1}{\ell!} \sum_{\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_\ell\in\mathcal{X}} \alpha(\vec{\mathbf{y}}_1)\cdots\alpha(\vec{\mathbf{y}}_\ell)\ \varphi(\vec{\mathbf{y}}_1)\cdots\varphi(\vec{\mathbf{y}}_\ell) \end{aligned}$$

Define the incidence graph $G(\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_\ell)$ to be the labelled graph with

- ▷ vertices $\{1, \cdots, \ell\}$ and
- ▷ an edge between $i \neq j$ when $\operatorname{supp} \vec{\mathbf{y}}_i \cap \operatorname{supp} \vec{\mathbf{y}}_j \neq \emptyset$.

For a subset of $Z \subset X$, denote by $\mathcal{C}(Z)$ the set of all ordered tuples $(\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_n)$ such that

- ▷ $Z = \operatorname{supp} \vec{\mathbf{y}}_1 \cup \cdots \cup \operatorname{supp} \vec{\mathbf{y}}_n$.
- ▷ $G(\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_n)$ is connected.

We call such a tuple a connected cover of Z.

So

$$\sum_{\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_\ell \in \mathcal{X}} \alpha(\vec{\mathbf{y}}_1) \cdots \alpha(\vec{\mathbf{y}}_\ell)\ \varphi(\vec{\mathbf{y}}_1) \cdots \varphi(\vec{\mathbf{y}}_\ell)$$

$$= \sum_{n=1}^{\ell} \frac{1}{n!} \sum_{\substack{Z_1, \cdots, Z_n \subset X \\ \text{pairwise disjoint} \\ \text{nonempty}}} \sum_{\substack{I_1 \cup \cdots \cup I_n = \{1, \cdots, \ell\} \\ I_1, \cdots, I_n \text{ pairwise disjoint}}} \sum_{\substack{\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_\ell \\ (\vec{\mathbf{y}}_i)_{i \in I_j} \in \mathcal{C}(Z_j)}} \alpha(\vec{\mathbf{y}}_1) \cdots \alpha(\vec{\mathbf{y}}_\ell)\ \varphi(\vec{\mathbf{y}}_1) \cdots \varphi(\vec{\mathbf{y}}_\ell)$$

Fix, for the moment, pairwise disjoint nonempty subsets $Z_1, \cdots, Z_n$ of X and $\ell \ge n$. Then

$$\sum_{\substack{I_1\cup\cdots\cup I_n=\{1,\cdots,\ell\}\\ I_1,\cdots,I_n \ \text{disjoint}}} \sum_{\substack{\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_\ell\\ (\vec{\mathbf{y}}_i,\, i\in I_j)\in\mathcal{C}(Z_j)}} \alpha(\vec{\mathbf{y}}_1)\cdots\varphi(\vec{\mathbf{y}}_\ell)$$

$$= \sum_{\substack{k_1,\cdots,k_n\ge 1\\ k_1+\cdots+k_n=\ell}} \sum_{\substack{I_1,\cdots,I_n\subset\{1,\cdots,\ell\}\\ I_1,\cdots,I_n \ \text{disjoint}\\ |I_j|=k_j}} \sum_{\substack{\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_\ell\\ (\vec{\mathbf{y}}_i,\, i\in I_j)\in\mathcal{C}(Z_j)}} \alpha(\vec{\mathbf{y}}_1)\cdots\varphi(\vec{\mathbf{y}}_\ell)$$

$$= \sum_{\substack{k_1,\cdots,k_n\ge 1\\ k_1+\cdots+k_n=\ell}} \frac{\ell!}{k_1!\cdots k_n!} \sum_{\substack{(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_{k_1})\in\mathcal{C}(Z_1)\\ \vdots\\ (\vec{\mathbf{y}}_{\ell-k_n+1},\cdots,\vec{\mathbf{y}}_\ell)\in\mathcal{C}(Z_n)}} \alpha(\vec{\mathbf{y}}_1)\cdots\varphi(\vec{\mathbf{y}}_\ell)$$

As the measure μ factorizes with each factor normalized, and the different Z_j's are disjoint,

$$\int \varphi(\vec{\mathbf{y}}_1)\cdots\varphi(\vec{\mathbf{y}}_\ell)\ d\mu(\varphi)$$
$$= \prod_{j=1}^{n} \int \varphi(\vec{\mathbf{y}}_{p_{j-1}+1})\cdots\varphi(\vec{\mathbf{y}}_{p_j})\ d\mu(\varphi)$$

(where $p_0 = 0$ and, for $1 \le j \le n$, $p_j = k_1 + \cdots + k_j$).

Writing

$$\int e^{F(\psi,\varphi)}\ d\mu(\varphi)$$

$$= 1 + \sum_{\ell=1}^{\infty} \tfrac{1}{\ell!} \sum_{n=1}^{\ell} \tfrac{1}{n!} \sum_{\substack{Z_1,\cdots,Z_n\subset X \\ \text{pairwise disjoint} \\ \text{nonempty}}} \sum_{\substack{k_1,\cdots,k_n\ge 1 \\ k_1+\cdots+k_n=\ell}} \frac{\ell!}{k_1!\cdots k_n!} \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \sum_{\ell=n}^{\infty} \tfrac{1}{n!} \sum_{\substack{Z_1,\cdots,Z_n\subset X \\ \text{pairwise disjoint} \\ \text{nonempty}}} \sum_{\substack{k_1,\cdots,k_n\ge 1 \\ k_1+\cdots+k_n=\ell}} \frac{1}{k_1!\cdots k_n!} \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{\substack{Z_1,\cdots,Z_n\subset X \\ \text{pairwise disjoint} \\ \text{nonempty}}} \sum_{k_1,\cdots,k_n\ge 1} \frac{1}{k_1!\cdots k_n!} \cdots$$

we have

$$\int e^{F(\psi,\varphi)}\ d\mu(\varphi) = 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{\substack{Z_1,\cdots,Z_n\subset X \\ \text{pairwise disjoint}}} \prod_{j=1}^{n} \Phi(Z_j)$$

where, for $\emptyset \ne Z \subset X$,

$$\Phi(Z) =$$

$$\sum_{k=1}^{\infty} \tfrac{1}{k!} \sum_{(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_k)\in\mathcal{C}(Z)} \alpha(\vec{\mathbf{y}}_1)\cdots\alpha(\vec{\mathbf{y}}_k) \int \varphi(\vec{\mathbf{y}}_1)\cdots\varphi(\vec{\mathbf{y}}_k)\ d\mu(\varphi) \tag{III.2}$$

and $\Phi(\emptyset) = 0$.

If we define

$$\zeta(Z,Z') = \begin{cases} 0 & \text{if } Z \cap Z' \neq \emptyset \\ 1 & \text{if } Z \text{ and } Z' \text{ are disjoint} \end{cases}$$

and $G_n = \{\ \{i,j\} \subset \mathbb{N}^2 \ \big|\ 1 \le i < j \le n\ \}$ is the complete graph on $\{1,\cdots,n\}$, then

$$\int e^{F(\psi,\varphi)}\ d\mu(\varphi)$$

$$= 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{Z_1,\cdots,Z_n \subset X} \prod_{\{i,j\}\in G_n} \zeta(Z_i,Z_j) \prod_{j=1}^{n} \Phi(Z_j)$$

$$= 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{Z_1,\cdots,Z_n} \Bigg(\sum_{g\subset G_n} \prod_{\{i,j\}\in g} \big(\zeta(Z_i,Z_j) - 1\big) \Bigg) \prod_{j=1}^{n} \Phi(Z_j)$$

$$= 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{Z_1,\cdots,Z_n \subset X} \rho(Z_1,\cdots,Z_n) \prod_{j=1}^{n} \Phi(Z_j)$$

where

$$\rho(Z_1,\cdots,Z_n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum\limits_{g\subset G_n} \prod\limits_{\{i,j\}\in g} \big(\zeta(Z_i,Z_j) - 1\big) & \text{if } n \ge 2 \end{cases}$$

Define

$$\rho^T(Z_1, \cdots, Z_n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum\limits_{g \in \mathcal{C}_n} \prod\limits_{\{i,j\} \in g} \big(\zeta(Z_i, Z_j) - 1\big) & \text{if } n \geq 2 \end{cases}$$

where $\mathcal{C}_n$ is the set of connected subgraphs of G_n. By a standard argument,

$$\ln \int e^{F(\psi,\varphi)} d\mu = \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{Z_1, \cdots, Z_n \subset X} \rho^T(Z_1, \cdots, Z_n) \prod_{j=1}^{n} \Phi(Z_j) \tag{III.3}$$

(By "$\ln$" we just mean that the exponential of the right hand side is $\int e^F(\psi, \varphi)\ d\mu$.)

Aside: Outline of the standard argument:

Define the value of the graph $g \subset G_n$ to be

$$\mathrm{Val}(g) = \begin{cases} \sum\limits_{Z \subset X} \Phi(Z) & \text{if } n = 1 \\ \sum\limits_{Z_1, \cdots, Z_n} \prod\limits_{\{i,j\} \in g} C(Z_i, Z_j) \prod\limits_{j=1}^{n} \Phi(Z_j) & \text{if } n > 1 \end{cases}$$

where $C(Z_i, Z_j) = \zeta(Z_i, Z_j) - 1$. If the connected components of $g \in \mathcal{G}_n$ are g_1, $\cdots$, g_m, then

$$\mathrm{Val}(g) = \prod_{j=1}^{m} \mathrm{Val}(g_m)$$

Consequently,

$$\begin{aligned} \int e^{F(\psi,\varphi)} d\mu &= 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{g \subset G_n} \mathrm{Val}(g) \\ &= \prod_{n=1}^{\infty} \prod_{g \in \mathcal{C}_n} e^{\frac{1}{n!} \mathrm{Val}(g)} \\ &= \exp\Big\{ \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{g \in \mathcal{C}_n} \mathrm{Val}(g) \Big\} \end{aligned}$$

End of aside.

We now find a, not necessarily symmetric, coefficient system for $\ln \int e^{F(\psi,\varphi)}\ d\mu(\varphi)$. Recall

$$\Phi(Z) = \sum_{k=1}^{\infty} \tfrac{1}{k!} \sum_{(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_k)\in\mathcal{C}(Z)} \alpha(\vec{\mathbf{y}}_1)\cdots\alpha(\vec{\mathbf{y}}_k)\int \varphi(\vec{\mathbf{y}}_1)\cdots\varphi(\vec{\mathbf{y}}_k)\ d\mu(\varphi) \tag{III.2}$$

and

$$\alpha(\vec{\mathbf{y}}) = \sum_{\vec{\mathbf{x}}\in\mathcal{X}} A(\vec{\mathbf{x}},\vec{\mathbf{y}})\ \psi(\vec{\mathbf{x}})$$

So, if we set, for each $(\vec{\mathbf{x}},\vec{\mathbf{y}}) \in \mathcal{X}^2$,

$$\tilde{A}(\vec{\mathbf{x}},\vec{\mathbf{y}}) = \sum_{k=1}^{\infty} \tfrac{1}{k!} \sum_{\substack{(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_k)\in\mathcal{C}(\mathrm{supp}\ \vec{\mathbf{y}})\\ \vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_k=\vec{\mathbf{y}}}} \sum_{\substack{\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_k\\ \vec{\mathbf{x}}_1\circ\cdots\circ\vec{\mathbf{x}}_k=\vec{\mathbf{x}}}} A(\vec{\mathbf{x}}_1,\vec{\mathbf{y}}_1)\cdots A(\vec{\mathbf{x}}_k,\vec{\mathbf{y}}_k)\int \varphi(\vec{\mathbf{y}})\ d\mu(\varphi) \tag{III.5}$$

Then

$$\Phi(Z)(\psi) = \sum_{\substack{(\vec{\mathbf{x}},\vec{\mathbf{y}})\in\mathcal{X}^2\\ \mathrm{supp}\ \vec{\mathbf{y}}=Z}} \tilde{A}(\vec{\mathbf{x}},\vec{\mathbf{y}})\ \psi(\vec{\mathbf{x}})$$

Recall that

$$\ln \int e^{F(\psi,\varphi)} d\mu = \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{Z_1,\cdots,Z_n \subset X} \rho^T(Z_1,\cdots,Z_n) \prod_{j=1}^{n} \Phi(Z_j) \tag{III.3}$$

Therefore,

$$\ln \int e^{F(\psi,\varphi)} \; d\mu(\varphi) = \sum_{\vec{\mathbf{x}}\in\mathcal{X}} a(\vec{\mathbf{x}}) \; \psi(\vec{\mathbf{x}})$$

where, for $\vec{\mathbf{x}} \in \mathcal{X}$,

$$a(\vec{\mathbf{x}}) = \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{\substack{\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_n\in\mathcal{X} \\ \vec{\mathbf{x}}_1\circ\cdots\circ\vec{\mathbf{x}}_n=\vec{\mathbf{x}}}} \sum_{\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_n\in\mathcal{X}} \rho^T(\mathrm{supp}\;\vec{\mathbf{y}}_1,\cdots,\mathrm{supp}\;\vec{\mathbf{y}}_n) \prod_{j=1}^{n} \tilde{A}(\vec{\mathbf{x}}_j,\vec{\mathbf{y}}_j) \tag{III.6}$$

Also

$$f(\psi) = \ln \frac{\int e^{F(\psi,\varphi)}\, d\mu(\varphi)}{\int e^{F(0,\varphi)}\, d\mu(\varphi)} = \sum_{\substack{\vec{\mathbf{x}}\in\mathcal{X} \\ n(\vec{\mathbf{x}})>0}} a(\vec{\mathbf{x}}) \; \psi(\vec{\mathbf{x}})$$

Norms

Definition II.6 (Norms for functions of one field)
Let

$$f(\psi) = \sum_{\vec{\mathbf{x}} \in \mathcal{X}} a(\vec{\mathbf{x}})\ \psi(\vec{\mathbf{x}})$$

with $a(\vec{\mathbf{x}})$ invariant under permutations of the components of $\vec{\mathbf{x}}$. We call $a = \left\{ a(\vec{\mathbf{x}}) \,\middle|\, \vec{\mathbf{x}} \in \mathcal{X} \right\}$ the symmetric coefficient system for f.
If $w(\vec{\mathbf{x}})$ is a weight system for one field, we define

$$\|f\|_w = \|a\|_w \equiv \sum_{n \geq 0} \max_{\substack{1 \leq i \leq n \\ \mathbf{z} \in X}} \sum_{\substack{\vec{\mathbf{x}} \in X^n \\ \mathbf{x}_i = \mathbf{z}}} w(\vec{\mathbf{x}})\ \left|a(\vec{\mathbf{x}})\right|$$

Here $\mathbf{x}_i$ is the $i^{\rm th}$ component of the n–tuple $\vec{\mathbf{x}}$. The term in the above sum with $n = 0$ is simply $w(-)\ \left|a(-)\right|$ where $-$ denotes the 0–tuple.

Remark II.7 If

$$f(\psi) = \sum_{\vec{\mathbf{x}} \in \mathcal{X}} a(\vec{\mathbf{x}})\ \psi(\vec{\mathbf{x}})$$

with $a(\vec{\mathbf{x}})$ **not necessarily** invariant under permutations of the components of $\vec{\mathbf{x}}$, then

$$\|f\|_w \le \|a\|_w \equiv \sum_{n \ge 0} \max_{\substack{1 \le i \le n \\ \mathbf{z} \in X}} \sum_{\substack{\vec{\mathbf{x}} \in X^n \\ \mathbf{x}_i = \mathbf{z}}} w(\vec{\mathbf{x}})\ \big|a(\vec{\mathbf{x}})\big|$$

Definition II.3 (Weight System for One Field) A weight system for one field is a function $w : \mathcal{X} \to (0, \infty)$ that satisfies:

(a) $w(\vec{\mathbf{x}})$ is invariant under permutations of the components of $\vec{\mathbf{x}}$.

(b) $$w(\vec{\mathbf{x}} \circ \vec{\mathbf{x}}') \le w(\vec{\mathbf{x}}) w(\vec{\mathbf{x}}')$$
for all $\vec{\mathbf{x}}, \vec{\mathbf{x}}' \in \mathcal{X}$ with $\operatorname{supp}(\vec{\mathbf{x}}) \cap \operatorname{supp}(\vec{\mathbf{x}}') \ne \emptyset$.

Example II.4 (Weight Systems)

(i) If $\kappa : X \to (0, \infty)$ (called a weight factor) then

$$w(\vec{\mathbf{x}}) = \kappa(\vec{\mathbf{x}}) = \prod_{\ell=1}^{n(\vec{\mathbf{x}})} \kappa(\mathbf{x}_\ell)$$

is a weight system for one field.

(ii) Let $d : X \times X \to \mathbb{R}_{\geq 0}$ be a metric. For a subset $S \subset X$, denote by $\tau(S)$ the length of the shortest tree in X whose set of vertices contains S. Then

$$w(\vec{\mathbf{x}}) = e^{\tau(\mathrm{supp}\,(\vec{\mathbf{x}}))}$$

is a weight system for one field.

(iii) If $w_1(\vec{\mathbf{x}})$ and $w_2(\vec{\mathbf{x}})$ are weight systems for one field, then so is

$$w_3(\vec{\mathbf{x}}) = w_1(\vec{\mathbf{x}}) w_2(\vec{\mathbf{x}})$$

Definition II.3 (Weight System for Two Fields) A weight system for two fields is a function $W : \mathcal{X}^2 \to (0,\infty)$ that satisfies:

(a) $W(\vec{\mathbf{x}},\vec{\mathbf{y}})$ is invariant under permutations of the components of $\vec{\mathbf{x}}$ and is invariant under permutations of the components of $\vec{\mathbf{y}}$.

(b) $$W(\vec{\mathbf{x}}\circ\vec{\mathbf{x}}',\vec{\mathbf{y}}\circ\vec{\mathbf{y}}') \le W(\vec{\mathbf{x}},\vec{\mathbf{y}})W(\vec{\mathbf{x}}',\vec{\mathbf{y}}')$$ whenever $\mathrm{supp}\,(\vec{\mathbf{x}},\vec{\mathbf{y}}) \cap \mathrm{supp}\,(\vec{\mathbf{x}}',\vec{\mathbf{y}}') \neq \emptyset$.

Definition II.6 (Norms for functions of two fields) Let

$$F(\psi,\varphi) = \sum_{(\vec{\mathbf{x}},\vec{\mathbf{y}})\in\mathcal{X}^2} A(\vec{\mathbf{x}},\vec{\mathbf{y}})\ \psi(\vec{\mathbf{x}})\varphi(\vec{\mathbf{y}})$$

with $A(\vec{\mathbf{x}},\vec{\mathbf{y}})$ invariant under permutations of the components of $\vec{\mathbf{x}}$ and under permutations of the components of $\vec{\mathbf{y}}$.

If $W(\vec{\mathbf{x}},\vec{\mathbf{y}})$ be a weight system for two fields, we define

$$\|F\|_W = \sum_{n,m\ge 0} \max_{\substack{1\le i\le n+m \\ \mathbf{z}\in X}} \sum_{\substack{(\vec{\mathbf{x}},\vec{\mathbf{y}})\in X^n\times X^m \\ (\vec{\mathbf{x}},\vec{\mathbf{y}})_i=\mathbf{z}}} W(\vec{\mathbf{x}},\vec{\mathbf{y}})\ \big|A(\vec{\mathbf{x}},\vec{\mathbf{y}})\big|$$

Here $(\vec{\mathbf{x}},\vec{\mathbf{y}})_i$ is the $i^{\rm th}$ component of the $n+m$–tuple $(\vec{\mathbf{x}},\vec{\mathbf{y}})$. The term in the above sum with $n=m=0$ is simply $W(-,-)\ \big|A(-,-)\big|$.

Review of the Main Theorem

Let $X(=\mathrm{space})$ be a finite set. Let $d\mu_0(t)$ be a normalized measure on $\mathbb{R}$ that is supported in $|t| \le r$ for some constant r. We endow $\mathbb{R}^X$ with the ultralocal product measure

$$d\mu(\varphi) = \prod_{\mathbf{x}\in X} d\mu_0\big(\varphi(\mathbf{x})\big)$$

Theorem III.4 *Let w and W be weight systems for 1 and 2 fields, respectively, that obey*

$$W(\vec{\mathbf{x}},\vec{\mathbf{y}}) \ge (4r)^{n(\vec{\mathbf{y}})} w(\vec{\mathbf{x}})$$

If $F(\psi,\varphi)$ obeys $\|F\|_W < \frac{1}{16}$, then there is a real analytic function $f(\psi)$ such that

$$\frac{\int e^{F(\psi,\varphi)}\, d\mu(\varphi)}{\int e^{F(0,\varphi)}\, d\mu(\varphi)} = e^{f(\psi)} \tag{III.1}$$

and

$$\|f\|_w \le \tfrac{\|F\|_W}{1-16\|F\|_W}$$

Outline of the Proof – Bounds

Step 1 - organizing the sums

Recall from (III.6) that

$$a(\vec{\mathbf{x}}) = \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{\substack{\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_n \in \mathcal{X} \\ \vec{\mathbf{x}}_1 \circ \cdots \circ \vec{\mathbf{x}}_n = \vec{\mathbf{x}}}} \sum_{\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_n \in \mathcal{X}} \rho^T(\text{supp } \vec{\mathbf{y}}_1, \cdots, \text{supp } \vec{\mathbf{y}}_n) \prod_{j=1}^{n} \tilde{A}(\vec{\mathbf{x}}_j, \vec{\mathbf{y}}_j)$$

The bound

$$\left|\rho^T(\text{supp } \vec{\mathbf{y}}_1, \cdots, \text{supp } \vec{\mathbf{y}}_n)\right| \le \#\{\text{ spanning trees in } G(\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_n) \ \}$$

is due to Rota.

Hence

$$|a(\vec{\mathbf{x}})| \le \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_n\in\mathcal{X} \\ \vec{\mathbf{x}}_1\circ\cdots\circ\vec{\mathbf{x}}_n=\vec{\mathbf{x}}}} \sum_{\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_n\in\mathcal{X}} \sum_{\substack{T \text{ spanning tree} \\ \text{for } G(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_n)}} \prod_{j=1}^{n} \left|\tilde{A}(\vec{\mathbf{x}}_j,\vec{\mathbf{y}}_j)\right|$$

$$\le \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{T \text{ labelled tree with} \\ \text{vertices } 1,\cdots,n}} \sum_{\vec{\mathbf{y}}\in\mathcal{X}} |\tilde{A}|_T(\vec{\mathbf{x}},\vec{\mathbf{y}}) \tag{III.8}$$

where

$$|\tilde{A}|_T(\vec{\mathbf{x}},\vec{\mathbf{y}}) = \sum_{\substack{\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_n\in\mathcal{X} \\ \vec{\mathbf{y}}=\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_n \\ T\subset G(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_n)}} \sum_{\substack{\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_n\in\mathcal{X} \\ \vec{\mathbf{x}}=\vec{\mathbf{x}}_1\circ\cdots\circ\vec{\mathbf{x}}_n}} \prod_{\ell=1}^{n} |\tilde{A}(\vec{\mathbf{x}}_\ell,\vec{\mathbf{y}}_\ell)|$$

Recall that

$$\tilde{A}(\vec{\mathbf{x}},\vec{\mathbf{y}}) = \sum_{k=1}^{\infty} \tfrac{1}{k!} \sum_{\substack{(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_k)\in\mathcal{C}(\text{supp } \vec{\mathbf{y}}) \\ \vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_k=\vec{\mathbf{y}}}} \sum_{\substack{\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_k \\ \vec{\mathbf{x}}_1\circ\cdots\circ\vec{\mathbf{x}}_k=\vec{\mathbf{x}}}} A(\vec{\mathbf{x}}_1,\vec{\mathbf{y}}_1)\cdots A(\vec{\mathbf{x}}_k,\vec{\mathbf{y}}_k) \int \varphi(\vec{\mathbf{y}})\ d\mu(\varphi) \tag{III.5}$$

For each $(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_k)$ $G(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_k)$ is connected and hence contains at least one tree. So

$$\begin{aligned} |\tilde{A}(\vec{\mathbf{x}},\vec{\mathbf{y}})| &\le \sum_{k=1}^{\infty} \tfrac{1}{k!} \sum_{\substack{T \text{ labelled tree} \\ \text{with vertices} \\ 1,\cdots,k}} \sum_{\substack{\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_k\in\mathcal{X} \\ \vec{\mathbf{y}}=\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_k \\ T\subset G(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_k)}} \sum_{\substack{\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_k\in\mathcal{X} \\ \vec{\mathbf{x}}=\vec{\mathbf{x}}_1\circ\cdots\circ\vec{\mathbf{x}}_k}} r^{n(\vec{\mathbf{y}})} \prod_{\ell=1}^{n} |A(\vec{\mathbf{x}}_\ell,\vec{\mathbf{y}}_\ell)| \\ &= \sum_{k=1}^{\infty} \tfrac{1}{k!} \sum_{\substack{T \text{ labelled tree} \\ \text{with vertices} \\ 1,\cdots,k}} r^{n(\vec{\mathbf{y}})} |A|_T(\vec{\mathbf{x}},\vec{\mathbf{y}}) \end{aligned} \tag{III.8$'$}$$

where

$$|A|_T(\vec{\mathbf{x}},\vec{\mathbf{y}}) = \sum_{\substack{\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_k\in\mathcal{X} \\ \vec{\mathbf{y}}=\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_k \\ T\subset G(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_k)}} \sum_{\substack{\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_n\in\mathcal{X} \\ \vec{\mathbf{x}}=\vec{\mathbf{x}}_1\circ\cdots\circ\vec{\mathbf{x}}_k}} \prod_{\ell=1}^{k} |A(\vec{\mathbf{x}}_\ell,\vec{\mathbf{y}}_\ell)|$$

Step 2 - bound on B_T

Lemma III.5 *Let ω be an arbitrary weight system for two fields and define the weight system ω' by*

$$\omega'(\vec{\mathbf{x}},\vec{\mathbf{y}}) = 2^{n(\vec{\mathbf{y}})}\omega(\vec{\mathbf{x}},\vec{\mathbf{y}})$$

Let T be a labelled tree with vertices $1,\cdots,n$ and coordination numbers $d_1,\cdots,d_n$. Let B be any (not necessarily symmetric) coefficient system for two fields with $B(-,-)=0$. We define a new coefficient system B_T by

$$B_T(\vec{\mathbf{x}},\vec{\mathbf{y}}) = \sum_{\substack{\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_n\in\mathcal{X}\\ \vec{\mathbf{y}}=\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_n\\ T\subset G(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_n)}} \sum_{\substack{\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_n\in\mathcal{X}\\ \vec{\mathbf{x}}=\vec{\mathbf{x}}_1\circ\cdots\circ\vec{\mathbf{x}}_n}} \prod_{\ell=1}^{n} B(\vec{\mathbf{x}}_\ell,\vec{\mathbf{y}}_\ell)$$

Then

$$\big\|B_T\big\|_\omega \le d_1!\cdots d_n!\ \|B\|_{\omega'}^n$$

Outline of proof:

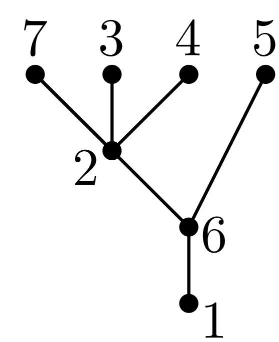
Ingredient 1:

- For each $1 \le \ell \le n$, think of $(\vec{\mathbf{x}}_\ell, \vec{\mathbf{y}}_\ell)$ as the locations of (two species of) stars in a galaxy.
- In computing

$$\big\|B_T\big\|_\omega = \sum_{N,M\ge 0} \max_{\substack{1\le i\le N+M \\ \mathbf{z}\in X}} \sum_{\substack{(\vec{\mathbf{x}},\vec{\mathbf{y}})\in X^N\times X^M \\ (\vec{\mathbf{x}},\vec{\mathbf{y}})_i=\mathbf{z}}} \omega(\vec{\mathbf{x}},\vec{\mathbf{y}})\ \big|B_T(\vec{\mathbf{x}},\vec{\mathbf{y}})\big|$$

  we must hold fixed the location of one star and sum over the locations of all other stars. Suppose that the fixed star is in galaxy $\ell = 1$.
- View 1 as the root of T.
- Then the set of vertices of T is endowed with a natural partial ordering under which 1 is the smallest vertex.
- For each vertex $2 \le \ell \le n$, denote by $\pi(\ell)$ the predecessor vertex of ℓ under this partial ordering.

$$\pi(7) = \pi(3) = \pi(4) = 2$$
$$\pi(2) = \pi(5) = 6$$
$$\pi(6) = 1$$



- ▷ The condition that $T \subset G(\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_n)$ ensures that, for each $2 \le \ell \le n$, the support of $\vec{\mathbf{y}}_\ell$ intersects the support of $\vec{\mathbf{y}}_{\pi(\ell)}$, so that at least one of the $n(\vec{\mathbf{y}}_\ell)$ components of $\vec{\mathbf{y}}_\ell$ takes the same value (in X) as some component of $\vec{\mathbf{y}}_{\pi(\ell)}$.
- ▷ Write $n(\vec{\mathbf{y}}_\ell) = n_\ell$.
- ▷ The product over $2 \le \ell \le n$ of the number of choices of which $\vec{\mathbf{y}}$–star in galaxy ℓ is at the same location of which $\vec{\mathbf{y}}$–star in galaxy $\pi(\ell)$ is

$$\prod_{\ell=2}^{n} \left[n_\ell n_{\pi(\ell)}) \right] = \prod_{\ell=1}^{n} n_\ell^{d_\ell} = \prod_{\ell=1}^{n} \frac{n_\ell^{d_\ell}}{d_\ell!} d_\ell! \\ \le d_1! \cdots d_n! \prod_{\ell=1}^{n} 2^{n_\ell}$$

by using first year calculus and Stirling.

Ingredient 2:

▷ Since T is connected,

$$\omega(\vec{\mathbf{x}},\vec{\mathbf{y}}) \le \prod_{\ell=1}^{n} \omega\big(\vec{\mathbf{x}}_\ell,\vec{\mathbf{y}}_\ell\big)$$

for all $\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_n \in \mathcal{X}$ and $\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_n \in \mathcal{X}$ under consideration. So we may absorb each factor $\omega\big(\vec{\mathbf{x}}_\ell,\vec{\mathbf{y}}_\ell\big)$ into $B\big(\vec{\mathbf{x}}_\ell,\vec{\mathbf{y}}_\ell\big)$ and it suffices to consider $\omega = 1$.

Ingredient 3:

▷ Iteratively apply

$$\sum_{\substack{\vec{\mathbf{x}}_\ell, \vec{\mathbf{y}}_\ell \in \mathcal{X} \\ \vec{\mathbf{y}}_{\ell, m_\ell} = \vec{\mathbf{y}}_{\pi(\ell), p_\ell}}} \sum_{\vec{\mathbf{x}}_\ell \in \mathcal{X}} 2^{n(\vec{\mathbf{y}}_\ell)} |B(\vec{\mathbf{x}}_\ell, \vec{\mathbf{y}}_\ell)| \le \|B\|_{\omega'}$$

starting with the largest ℓ's, in the partial ordering of T, and ending with $\ell = 1$. (For $\ell = 1$, substitute $\vec{\mathbf{x}}_{1,1} = \mathbf{x}$ for $\vec{\mathbf{y}}_{\ell, m_\ell} = \vec{\mathbf{y}}_{\pi(\ell), p_\ell}$.)

■

Step 3 - sum over n and T

Lemma III.6 *Let* $0 < \varepsilon < \frac{1}{8}$. *Then*

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{d_1,\cdots,d_n \\ d_1+\cdots+d_n=2(n-1)}} \sum_{\substack{T \text{ labelled tree} \\ \text{with coordination} \\ \text{numbers } d_1,\cdots,d_n}} d_1! \cdots d_n! \, \varepsilon^n$$

$$\le \frac{\varepsilon}{1-8\varepsilon}$$

Proof: By Cayley, the number of labelled trees on $n \ge 2$ vertices with coordination numbers $(d_1, d_2, \cdots, d_n)$ is

$$\frac{(n-2)!}{\prod_{j=1}^{n}(d_j-1)!}$$

The number of possible choices of coordination numbers $(d_1, d_2, \cdots, d_n) \in \mathbb{N}^n$ subject to the given constraint is

$$\binom{2(n-1)-1}{n-1} = \binom{2n-3}{n-1} \le 2^{2n-3}$$

Therefore

$$\sum_{n=2}^{\infty} \frac{1}{n!} \sum_{\substack{d_1,\cdots,d_n \\ d_1+\cdots+d_n=2(n-1)}} \sum_{\substack{T \text{ labelled tree} \\ \text{with coordination} \\ \text{numbers } d_1,\cdots,d_n}} d_1!\cdots d_n!\, \varepsilon^n$$

$$\le \sum_{n=2}^{\infty} \sum_{\substack{d_1,\cdots,d_n \\ d_1+\cdots+d_n=2(n-1)}} d_1 \cdots d_n\ \varepsilon^n$$

$$\le \sum_{n=2}^{\infty} 2^{2n-3}\, 2^n \varepsilon^n \ = \ \frac{8\varepsilon^2}{1-8\varepsilon}$$

For $n = 1$, $d_1 = 0$ and the number of trees is 1, so the $n = 1$ term is ε. So the full sum is bounded by $\varepsilon + \frac{8\varepsilon^2}{1-8\varepsilon} = \frac{\varepsilon}{1-8\varepsilon}$.

■

Step 4 - bound on $\|a\|$ in terms of $\|\tilde{A}\|$

We introduce, for each $\sigma > 0$, the auxiliary weight system

$$W_\sigma(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = W(\vec{\mathbf{x}}, \vec{\mathbf{y}}) \, \left(\tfrac{\sigma}{4r}\right)^{n(\vec{\mathbf{y}})}$$

Clearly

$$W_{4r}(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = W(\vec{\mathbf{x}}, \vec{\mathbf{y}}) \quad \text{and} \quad w(\vec{\mathbf{x}}) \leq W_1(\vec{\mathbf{x}}, \vec{\mathbf{y}}) \tag{III.9}$$

for all $(\vec{\mathbf{x}}, \vec{\mathbf{y}}) \in \mathcal{X}^2$.

We now prove

$$\|a\|_w \leq \frac{\|\tilde{A}\|_{W_2}}{1 - 8\|\tilde{A}\|_{W_2}} \tag{III.10}$$

Recall from (III.8) that

$$|a(\vec{\mathbf{x}})| \leq \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{\substack{T \text{ labelled tree with} \\ \text{vertices } 1,\cdots,n}} \sum_{\vec{\mathbf{y}} \in \mathcal{X}} |\tilde{A}|_T(\vec{\mathbf{x}}, \vec{\mathbf{y}})$$

Therefore, by (III.9) and Lemma III.5, with $\omega = W_1$ and $\omega' = W_2$,

$$\|a\|_w \le \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{T \text{ labelled tree with} \\ \text{vertices } 1,\cdots,n}} \big\| \, |\tilde{A}|_T \big\|_{W_1}$$

$$\le \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{d_1,\cdots,d_n \\ d_1+\cdots+d_n=2(n-1)}} \sum_{\substack{T \text{ labelled tree} \\ \text{with coordination} \\ \text{numbers } d_1,\cdots,d_n}} \big\| \, |\tilde{A}|_T \big\|_{W_1}$$

$$\le \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{d_1,\cdots,d_n \\ d_1+\cdots+d_n=2(n-1)}} \sum_{\substack{T \text{ labelled tree} \\ \text{with coordination} \\ \text{numbers } d_1,\cdots,d_n}} d_1! \cdots d_n! \, \big\| |\tilde{A}| \big\|_{W_2}^n$$

Now apply Lemma III.6 with $\varepsilon = \big\| |\tilde{A}| \big\|_{W_2} = \|\tilde{A}\|_{W_2}$ to get

$$\|a\|_w \le \frac{\|\tilde{A}\|_{W_2}}{1-8\|\tilde{A}\|_{W_2}}$$

Step 5 - bound on $\|\tilde{A}\|$ in terms of $\|A\|$

We now prove

$$\|\tilde{A}\|_{W_2} \le \tfrac{\|A\|_W}{1-8\|A\|_W} = \tfrac{\|F\|_W}{1-8\|F\|_W} \qquad \text{(III.11)}$$

Note that combining (III.10) and (III.11) yields the final bound

$$\|f\|_w \le \frac{\|\tilde{A}\|_{W_2}}{1-8\|\tilde{A}\|_{W_2}} \le \frac{\frac{\|F\|_W}{1-8\|F\|_W}}{1-8\frac{\|F\|_W}{1-8\|F\|_W}} = \frac{\|F\|_W}{1-16\|F\|_W}$$

Recall from (III.8') that

$$|\tilde{A}(\vec{\mathbf{x}},\vec{\mathbf{y}})| \le \sum_{k=1}^{\infty} \tfrac{1}{k!} \sum_{\substack{T \text{ labelled tree with} \\ \text{vertices } 1,\cdots,k}} r^{n(\vec{\mathbf{y}})} |A|_T(\vec{\mathbf{x}},\vec{\mathbf{y}})$$

By construction, $\left\| r^{n(\vec{\mathbf{y}})} |A|_T(\vec{\mathbf{x}},\vec{\mathbf{y}}) \right\|_{W_2} = \left\| |A|_T \right\|_{W_{2r}}$. Hence, by Lemma III.5, with $\omega = W_{2r}$ followed by Lemma III.6,

$$\|\tilde{A}\|_{W_2} \le \sum_{k=1}^{\infty} \tfrac{1}{k!} \sum_{\substack{T \text{ labelled tree with} \\ \text{vertices } 1,\cdots,k}} \left\| |A|_T \right\|_{W_{2r}}$$

$$\le \sum_{k=1}^{\infty} \tfrac{1}{k!} \sum_{\substack{d_1,\cdots,d_k \\ d_1+\cdots+d_k=2(k-1)}} \sum_{\substack{T \text{ labelled tree} \\ \text{with coordination} \\ \text{numbers } d_1,\cdots,d_k}} d_1!\cdots d_k! \; \|A\|_{W_{4r}}^k$$

$$\le \tfrac{\|A\|_W}{1-8\|A\|_W}$$

since $W_{4r} = W$. This gives (III.11). ■

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