Error Behaviour - A Trivial Example

In these notes, we look at the error that results when various numerical methods are used to generate approximate solutions to the initial value problem

\[ \begin{align*}
y' &= y \\
y(0) &= 1
\end{align*} \]

This problem is so simple that we can determine exactly both the real solution

\[ \phi(t) = e^t \]

and the approximate solution generated by Euler, Improved Euler and Runge-Kutta.

Euler’s Method
Since \( f(t, y) = y \), Euler’s method with step size \( h \) says

\[ y_{n+1} = y_n + hf(t_n, y_n) = y_n + hy_n = (1 + h)y_n \]

Hence

\[ \begin{align*}
y_0 &= 1 \\
y_1 &= (1 + h)y_0 = 1 + h \\
y_2 &= (1 + h)y_1 = (1 + h)^2 \\
& \vdots \\
y_n &= (1 + h)^n
\end{align*} \]

The number \( y_n \) is the value of the approximate solution at \( t = nh \), so the error \( E(h, t) \) at time \( t \) with step size \( h \) (assuming that \( t \) is an integer multiple of \( h \)) is

\[ E(h, t) = \phi(t) - y_{t/h} = e^t - (1 + h)^{t/h} = e^t - e^{\frac{t}{h} \ln(1+h)} \]

Assuming that \( h \ll 1 \), then by Taylor’s expansion

\[ \frac{1}{h} \ln(1 + h) = \frac{1}{h} \left( h - \frac{1}{2}h^2 + O(h^3) \right) = 1 - \frac{1}{2}h + O(h^2) \]

The notation \( O(h^n) \) is used to denote a function whose absolute value is bounded by a constant times \( h^n \) when \( h \) is sufficiently small. Subbing in

\[ E(h, t) = e^t - e^{t - th/2 + O(h^2 t)} \]
If \( th \ll 1 \) (or equivalently \( t^2/n \ll 1 \) or \( n \gg t^2 \)) in addition to \( h \ll 1 \)

\[
E(h, t) = e^t - e^{t-th/2 + O(h^2t)}
= e^t \left( 1 - e^{-th/2 + O(h^2t)} \right)
= e^t \left( 1 - 1 + th/2 + O(h^2t) + O(t^2h^2) \right)
\]

We have just substituted in \( e^x = 1 + x + O(x^2) \) and used \( h^2t \leq ht \) to simplify the \( O(x^2) \) term. So the error

\[
E(h, t) = \frac{t}{2} e^t h (1 + O(h))
\]

grows linearly with \( h \) and exponentially with \( t \).

**Improved Euler Method**

Since \( f(t, y) = y \), the Improved Euler method with step size \( h \) says

\[
y_{n+1} = y_n + \frac{h}{2} \left( f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n)) \right)
= y_n + \frac{h}{2} (y_n + y_n + hf(t_n, y_n))
= y_n + \frac{h}{2} (y_n + y_n + hy_n)
= y_n \left[ 1 + \frac{h}{2} (2 + h) \right]
\]

Iterating

\[
y_n = \left( 1 + h + \frac{h^2}{2} \right)^n = \left( 1 + h + \frac{h^2}{2} \right)^{t/h}
\]

so that the error is

\[
E(h, t) = \phi(t) - y_{t/h}
= e^t - \left( 1 + h + \frac{h^2}{2} \right)^{t/h}
= e^t - e^t \frac{\ln(1 + h + h^2/2)}{h}
\]

Taylor expanding

\[
\ln(1 + h + h^2/2) = h + \frac{1}{2} h^2 - \frac{1}{2} (h + \frac{1}{2} h^2)^2 + \frac{1}{3} (h + \frac{1}{2} h^2)^3 + O(h^4)
= h + \frac{1}{2} h^2 - \frac{1}{2} h^2 - \frac{1}{2} h^3 + O(h^4) + \frac{1}{3} h^3 + O(h^4)
= h - \frac{1}{6} h^3 + O(h^4)
\]

and Taylor expanding again

\[
E(h, t) = e^t - e^t \frac{\ln(h-h^3/6 + O(h^4))}{h} = e^t - e^t (1 - h^2/6 + O(h^3))
= e^t \left( 1 - e^{-th^2/6 + O(th^3)} \right)
= e^t \left( 1 - 1 - [-th^2/6 + O(th^3)] + O(t^2h^4) \right)
= \frac{1}{6} te^t h^2 \left( 1 + O(h) + O(th^2) \right)
\]
This time the error is proportional to $h^2$.

**Runge Kutta**

Since $f(t, y) = y$, 4th order Runge Kutta with step size $h$ says

\[ k_{n1} = f(t_n, y_n) = y_n \]
\[ k_{n2} = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_{n1}) = y_n + \frac{h}{2}k_{n1} = y_n \left(1 + \frac{h}{2}\right) \]
\[ k_{n3} = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_{n2}) = y_n + \frac{h}{2}k_{n2} = y_n \left(1 + \frac{h}{2} + \frac{h^2}{4}\right) \]
\[ k_{n4} = f(t_n + h, y_n + hk_{n3}) = y_n + hk_{n3} = y_n \left(1 + h + \frac{h^2}{2} + \frac{h^3}{4}\right) \]
\[ y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}) \]

Since

\[ \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}) = \frac{h}{6}y_n \left(1 + 2 + h + 2 + h + \frac{h^2}{2} + 1 + h + \frac{h^2}{2} + \frac{h^3}{4}\right) = \frac{h}{6}y_n \left(6 + 3h + h^2 + \frac{h^3}{4}\right) \]

we iterate

\[ y_{n+1} = y_n \left(1 + h + \frac{1}{2}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right) \]

to yield

\[ y_n = \left(1 + h + \frac{1}{2}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right)^n = \left(1 + h + \frac{1}{2}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right)^{t/h} \]

so that the error is

\[ E(h, t) = \phi(t) - y_{t/h} = e^t - \left(1 + h + \frac{1}{2}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right)^{t/h} \]

We can of course expand in powers of $h$ just like we did with Euler and Improved Euler. But it is easier to use a computer algebra package like maple. The maple commands

\[ \text{err} := (h, t) \rightarrow e^{-t} (1 + h + h^2/2 + h^3/6 + h^4/24)^t/h; \]
\[ \text{series}(\text{err}(h, t), h = 0, 6); \]

first defines the function err$(h, t)$ and then computes its Taylor expansion in $h$ about $h = 0$ to order 6. The result is

\[ 1 - \frac{1}{120}th^4 + \frac{1}{144}th^5 + O(h^6) \]

so that the error

\[ E(h, t) = \frac{1}{120}te^t h^4 (1 + O(h)) \]

is of order $h^4$. 

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