

Inverse Spectral Problems

The displacement, $y(x, t)$, at position x and time t of a vibrating string of length L , tension T and variable density, $\rho(x)$, that is held fixed at $x = 0$ and $x = L$ obeys the wave equation

$$\rho(x) \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

with boundary conditions $y(0, t) = y(L, t) = 0$ (assuming that the vibrations are small). For solutions of fixed (temporal) frequency, $y(x, t) = \text{Re} [v(x)e^{-i\omega t}]$ with

$$Tv'' + \omega^2 \rho(x)v = 0 \quad v(0) = v(L) = 0$$

This is not quite in the standard form of an eigenvalue problem because of the factor $\rho(x)$. We can turn it into the standard form, while keeping the coefficient of the second derivative term constant, by making a substitution of the form $v(x) = \alpha(x)u(z(x))$. This gives

$$-u''(x) + q(x)u(x) = \lambda u(x) \quad u(0) = u(1) = 0$$

There is a nontrivial u satisfying these conditions only for a discrete set of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

The inverse problem that corresponds to the direct problem of finding the spectrum $\{\lambda_1, \lambda_2, \lambda_3, \dots\}$ for a specified potential $q(x)$ is the

Question: Does the spectrum determine q ?

To get a first hint concerning the answer let's look at what happens when q is really small. If q is identically zero (i.e. ρ is constant), the normalized eigenfunctions are $u_n(x) = \sqrt{2} \sin(n\pi x)$ and the eigenvalues are $\lambda_n = n^2\pi^2$. (The corresponding values of ω are $\omega_n = \sqrt{\frac{T}{\rho}} \frac{n\pi}{L}$ – the famous fundamental and harmonic frequencies.) Let $\lambda_n(q)$ denote the n^{th} eigenvalue for potential q . It is an easy matter to determine not only $\lambda_n(0)$, but also

$$\frac{d}{dt} \lambda_n(tq) \Big|_{t=0} = \int_0^1 dx q(x) 2 \sin^2(n\pi x) = \int_0^1 dx q(x) [1 - \cos(2n\pi x)]$$

One can also compute higher order derivatives of λ_n at $q = 0$, but they are not of interest here. So

$$\lambda_n(q) = n^2\pi^2 + \int_0^1 dx q(x) - \int_0^1 dx q(x) \cos(2n\pi x) + O(q^2)$$

If we know that q is really small, then measuring the spectrum, and ignoring errors of order q^2 , gives us all of the even Fourier coefficients of q and none of the odd Fourier coefficients. Actually, that the spectrum does not determine q ought not be a surprise.

Answer: Obviously, the spectrum does not determine q because $q(x)$ and $q(1-x)$ always give the same spectrum.

Of course that is just the beginning of the story. In this one dimensional setting, one can say a lot more:

- The set of all spectra of real potentials $q \in L^2(0, 1)$ is exactly the set of all real, strictly increasing sequences of the form

$$\sigma_n = n^2\pi^2 + s + \tilde{\sigma}_n, \quad n \geq 1 \quad \text{with} \quad s \in \mathbb{R} \quad \text{and} \quad (\tilde{\sigma}_n)_{n \in \mathbb{N}} \in \ell^2$$

- If q is even (i.e. obeys $q(x) = q(1-x)$), then the spectrum determines q . That is, if $p, q \in L^2(0, 1)$, $p(1-x) = p(x)$, $q(1-x) = q(x)$ and $\lambda_n(p) = \lambda_n(q)$ for all $n \in \mathbb{N}$, then $p = q$.
- One can say a lot about the isospectral sets

$$M(p) = \{ q \in L^2(0, 1) \mid \lambda_n(q) = \lambda_n(p) \quad \forall n \in \mathbb{N} \}$$

In particular, the even element of $M(p)$ is always the element of $M(p)$ of smallest L^2 norm. See the book of Kirsch and the book of Pöschel and Trubowitz.

If we increase the dimension, the problem becomes a lot harder. Let $\Omega \subset \mathbb{R}^2$ be simply connected and compact and have a piecewise smooth boundary. Stretch a membrane over Ω and fix it to the boundary, $\partial\Omega$, of Ω while leaving it free to vibrate in the interior of Ω . Let $F(x, y, t)$ denote the displacement (in the direction perpendicular to $\mathbb{R}^2 \subset \mathbb{R}^3$) of the membrane at position (x, y) at time t . Assuming the mass density and tension of the membrane are uniform and that the vibrations are small,

$$\frac{\partial^2 F}{\partial t^2} = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) F \quad \text{in } \Omega, \quad F = 0 \quad \text{on } \partial\Omega$$

where c depends on the tension and mass density of the membrane. Choose units so that $c^2 = \frac{1}{2}$. For solutions of fixed (temporal) frequency, $F(x, t) = \text{Re} [U(x, y)e^{i\omega t}]$ with

$$\frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) U(x, y) + \omega^2 U(x, y) = 0 \quad \text{in } \Omega, \quad U = 0 \quad \text{on } \partial\Omega$$

Since Ω is compact, the problem

$$\frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) U(x, y) + \lambda U(x, y) = 0 \quad \text{in } \Omega, \quad U = 0 \quad \text{on } \partial\Omega$$

has a nontrivial (i.e. U not identically zero) solution only for a discrete family

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

of λ 's, which is called the spectrum of $-\frac{1}{2}\Delta$ with Dirichlet boundary conditions on $\partial\Omega$. Once again we have the

Question: Does the spectrum determine Ω ?

Once again the short answer is obvious.

Answer: No way – if you translate or rotate or reflect Ω , you do not change the spectrum.

So we refine the question.

Question: Given the spectrum, can you determine Ω up to Euclidean transformation?

This question was the subject of a famous and beautiful paper by Mark Kac entitled “Can One Hear the Shape of a Drum?” At the time of Kac’s article, the answer was unknown. But one did know that

- 1) The spectrum does determine the area and perimeter of Ω . One way to see this is to prove that the “heat trace”

$$\text{tr } e^{\frac{1}{2}\Delta t} = \sum_{n=1}^{\infty} e^{-\lambda_n t} = \frac{|\Omega|}{2\pi t} - \frac{L}{4\sqrt{2\pi t}} + o\left(\frac{1}{\sqrt{t}}\right) \text{ as } t \rightarrow 0$$

- 2) The isoperimetric inequality says that $L^2 \geq 4\pi|\Omega|$ with equality only for a circle. Hence if the spectrum $\lambda_1, \lambda_2, \dots$ is that of a circular drum, then Ω must be a circle.

Recently,

- 3) Zelditch has proven that the map from domains to spectra is one-to-one on a large class of bounded, simply connected, domains with analytic boundaries and one axis of symmetry.

This is strongly suggestive. But wrong. The answer is now known to be “no”. Gordon, Webb and Wolpert constructed isospectral drums by gluing together triangles in different ways, as in the figure on the last page. There “ $-B + C - E$ ”, for example, instructs you to take the specified linear combination of the B , C and E portions of the eigenfunction for the drum on the left. So we have the

Answer: Sometimes the spectrum determines the shape of Ω . Sometimes it doesn’t.

This is not very satisfying. But there is a related problem where the global picture is clearer. For two-dimensional manifolds (without boundary):

- Generically, the spectrum of the Laplace–Beltrami operator determines the manifold.
- The map from manifolds to spectra is always finite-to-one.

References

- C. Gordon, D. Webb, and S. Wolpert. **One cannot hear the shape of a drum**, Bull. Amer. Math. Soc., **27**, 134-138 (1992).
- Mark Kac, **Can One Hear the Shape of a Drum?**, American Mathematical Monthly, **73**, Issue 4, Part 2, 1-23 (1966).
- Andreas Kirsch, **An Introduction to the Mathematical Theory of Inverse Problems**, Springer, 1996.
- Jürgen Pöschel and Eugene Trubowitz, **Inverse Spectral Theory**, Academic Press, 1987.
- Steve Zelditch, **Inverse spectral problem for analytic plane domains II: domains with one symmetry**, preprint.

Isospectral Drums

