

# Inverse Scattering

Suppose that we are interested in a system in which sound waves, for example, scatter off of some obstacle. Let  $p(\mathbf{x}, t)$  be the pressure at position  $\mathbf{x}$  and time  $t$ . In (a somewhat idealized) free space,  $p$  obeys the wave equation  $\frac{\partial^2 p}{\partial t^2} = c^2 \Delta p$ , where  $c$  is the speed of sound. We shall assume that in most of the world,  $c$  takes a constant value  $c_0$ . But we introduce an obstacle by allowing  $c$  to depend on position in some compact region. We further allow for some absorption in that region. Then  $p$  obeys

$$\frac{\partial^2 p}{\partial t^2} + \gamma(\mathbf{x}) \frac{\partial p}{\partial t} = c(\mathbf{x})^2 \Delta p$$

where  $\gamma(\mathbf{x})$  is the damping coefficient of the medium at  $\mathbf{x}$ . For solutions of fixed (temporal) frequency,  $p(\mathbf{x}, t) = \text{Re} [u(\mathbf{x})e^{-i\omega t}]$  with

$$\Delta u + \frac{\omega^2}{c(\mathbf{x})^2} \left[1 + i \frac{\gamma(\mathbf{x})}{\omega}\right] u = 0$$

Outside of some compact region

$$\frac{\omega^2}{c(\mathbf{x})^2} \left[1 + i \frac{\gamma(\mathbf{x})}{\omega}\right] = \frac{\omega^2}{c_0^2} = k^2 \quad \text{where} \quad k = \frac{\omega}{c_0} > 0$$

If we define the index of refraction by

$$n(\mathbf{x}) = \frac{c_0^2}{c(\mathbf{x})^2} \left[1 + i \frac{\gamma(\mathbf{x})}{\omega}\right]$$

then

$$\Delta u + k^2 n(\mathbf{x}) u = 0 \tag{1}$$

with  $n(\mathbf{x}) = 1$  outside of some compact region. We first consider two special cases.

**Example 1 (Free Space)** In the absence of any obstacle  $\Delta u + k^2 u = 0$  on all of  $\mathbb{R}^3$ . Then we can solve just by Fourier transforming. The general solution is a mixture of solutions of the form  $u = e^{ik\hat{\boldsymbol{\theta}} \cdot \mathbf{x}}$  where  $\hat{\boldsymbol{\theta}}$  is a unit vector. This represents a plane wave coming in from infinity in direction  $\hat{\boldsymbol{\theta}}$ .

**Example 2 (Point Source)** If we have free space everywhere except at the origin and we have a unit point source at the origin, then

$$\Delta u + k^2 u = \delta(\mathbf{x})$$

Except at the origin, where there is a singularity, we still have  $\Delta u + k^2 u = 0$ . The point source generates expanding spherical waves. So  $u$  should be a function of  $r = |\mathbf{x}|$  only and obey

$$u''(r) + \frac{2}{r}u'(r) + k^2u(r) = 0$$

This is easily solved by changing variables to  $v(r) = ru(r)$ , which obeys

$$v''(r) + k^2v(r) = 0$$

So  $v(r) = \alpha \sin(kr) + \beta \cos(kr)$  and  $u(r) = \alpha \frac{\sin(kr)}{r} + \beta \frac{\cos(kr)}{r}$ . To be an outgoing (rather than incoming) wave  $u(r) = \alpha' \frac{e^{ikr}}{r}$ . (Note that  $e^{ikr} e^{-i\omega t}$  is constant on  $r = \frac{\omega}{k}t$ , which is a sphere that is expanding with speed  $c_0$ .) To give the Dirac delta function on the right hand side of  $\Delta u + k^2 u = \delta(\mathbf{x})$  coefficient one, we need  $u(\mathbf{x}) = -\frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}$ . (See, for example, the notes on Poisson's equation.)

Now let's return to the general case. We want to think of a physical situation in which we send a plane wave  $u^i(\mathbf{x}) = e^{ik\hat{\boldsymbol{\theta}} \cdot \mathbf{x}}$  in from infinity. This plane wave shakes up the obstacle which then emits a bunch of expanding spherical waves  $\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$  emanating from various points  $\mathbf{y}$  in the obstacle. So the full solution is of the form

$$u(\mathbf{x}) = u^i(\mathbf{x}) + u^s(\mathbf{x})$$

where the scattered wave,  $u^s$ , obeys the "radiation condition"

$$\frac{\partial}{\partial r} u^s(\mathbf{x}) - ik u^s(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|^2}\right) \quad \text{as} \quad |\mathbf{x}| \rightarrow \infty \quad (2)$$

This condition is chosen to allow outgoing waves  $\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$  but not incoming waves  $\frac{e^{-ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$ .

Define

$$\Phi(\mathbf{x}, \mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$$

Since  $\delta(\mathbf{x} - \mathbf{y})$  is the kernel of the identity operator,

$$(\Delta_x + \mathbf{k}^2)\Phi(\mathbf{x}, \mathbf{y}) = -\delta(\mathbf{x} - \mathbf{y})$$

says, roughly, that  $u(\mathbf{x}) \mapsto -\int \Phi(\mathbf{x}, \mathbf{y})u(\mathbf{y}) d\mathbf{y}$  is the inverse of the map  $u(\mathbf{x}) \mapsto (\Delta + k^2)u(\mathbf{x})$  for functions that obey the radiation condition. We can exploit this to convert (1), (2) into an equivalent integral equation

$$\begin{aligned} \Delta u + k^2 n(\mathbf{x})u = 0 &\implies \Delta u + k^2 u = k^2(1 - n(\mathbf{x}))u \\ &\implies \Delta u^s + k^2 u^s = k^2(1 - n(\mathbf{x}))u \end{aligned}$$

since  $\Delta u^i + k^2 u^i = 0$ . As  $u^s$  obeys the radiation condition

$$u^s(\mathbf{x}) = -k^2 \int \Phi(\mathbf{x}, \mathbf{y})(1 - n(\mathbf{y}))u(\mathbf{y}) d\mathbf{y}$$

so that

$$u(\mathbf{x}) = u^i(\mathbf{x}) - k^2 \int (1 - n(\mathbf{y}))\Phi(\mathbf{x}, \mathbf{y})u(\mathbf{y}) d\mathbf{y} \quad (3)$$

This is called the Lippmann–Schwinger equation. Observe that it is of the form  $u = u^i - Fu$  or  $(\mathbb{1} - F)u = u^i$  where  $F$  is the linear operator  $u(\mathbf{x}) \mapsto k^2 \int \Phi(\mathbf{x}, \mathbf{y})(1 - n(\mathbf{y}))u(\mathbf{y}) d\mathbf{y}$ . This operator is compact (if you impose the appropriate norms) and so behaves much like a finite dimensional matrix. If  $F$  has operator norm smaller than one, which is the case if  $k^2(1 - n)$  is small enough, then  $\mathbb{1} - F$  is trivially invertible and the equation  $(\mathbb{1} - F)u = u^i$  has a unique solution. Even if  $F$  has operator norm larger than or equal to one,  $(\mathbb{1} - F)u = u^i$  fails to have a unique solution only if  $F$  has eigenvalue one. One can show that this is impossible in the present setting. Thus, one can prove

**Theorem.** *If  $n \in C^2(\mathbb{R}^3)$ ,  $n(\mathbf{x}) - 1$  has compact support and  $\operatorname{Re} n(\mathbf{x}), \operatorname{Im} n(\mathbf{x}) \geq 0$ , then (1), (2) has a unique solution.*

For large  $|\mathbf{x}|$ ,  $\Phi$  has the asymptotic behaviour

$$\Phi(\mathbf{x}, \mathbf{y}) = \frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|} e^{-ik\hat{\mathbf{x}} \cdot \mathbf{y}} + O\left(\frac{1}{|\mathbf{x}|^2}\right)$$

so that, when the incoming plane wave is moving in direction  $\hat{\boldsymbol{\theta}}$ ,

$$u(\mathbf{x}; \hat{\boldsymbol{\theta}}) = u^i(\mathbf{x}; \hat{\boldsymbol{\theta}}) + \frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|} u_\infty(\hat{\mathbf{x}}; \hat{\boldsymbol{\theta}}) + O\left(\frac{1}{|\mathbf{x}|^2}\right)$$

where

$$u_\infty(\hat{\mathbf{x}}; \hat{\boldsymbol{\theta}}) = -k^2 \int e^{-ik\hat{\mathbf{x}} \cdot \mathbf{y}} (1 - n(\mathbf{y}))u(\mathbf{y}; \hat{\boldsymbol{\theta}}) d\mathbf{y}$$

If we are observing the scattered wave from vantage points far from the obstacle, we will only be able to measure  $u_\infty(\hat{\mathbf{x}}; \hat{\boldsymbol{\theta}})$ . The inverse problem then is

**Question:** Given  $u_\infty(\hat{\mathbf{x}}; \hat{\boldsymbol{\theta}})$ , for all  $\hat{\mathbf{x}}, \hat{\boldsymbol{\theta}} \in S^2$ , can we determine  $n$ ? The short answer is

**Answer:** Yes, because we have the

**Theorem.** *If  $n_1, n_2 \in C^2(\mathbb{R}^3)$  with  $n_1 - 1, n_2 - 1$  of compact support and  $u_{1,\infty}(\hat{\mathbf{x}}; \hat{\boldsymbol{\theta}}) = u_{2,\infty}(\hat{\mathbf{x}}; \hat{\boldsymbol{\theta}})$ , for all  $\hat{\mathbf{x}}, \hat{\boldsymbol{\theta}} \in S^2$ , then  $n_1 = n_2$ .*

We can get a rough idea why this Theorem is true by looking at the Born approximation. In this approximation  $u^s$  is ignored in the computation of  $u_\infty$  so that

$$\begin{aligned} u_\infty(\hat{\mathbf{x}}; \hat{\boldsymbol{\theta}}) &\approx -k^2 \int e^{-ik\hat{\mathbf{x}} \cdot \mathbf{y}} (1 - n(\mathbf{y}))u^i(\mathbf{y}; \hat{\boldsymbol{\theta}}) d\mathbf{y} \\ &= -k^2 \int e^{-ik(\hat{\mathbf{x}} - \hat{\boldsymbol{\theta}}) \cdot \mathbf{y}} (1 - n(\mathbf{y})) d\mathbf{y} \end{aligned}$$

If we measure  $u_\infty(\hat{\mathbf{x}}; \hat{\boldsymbol{\theta}})$ , then, in this approximation, we know the Fourier transform of  $1 - n(\mathbf{y})$  on the set  $\{ k(\hat{\mathbf{x}} - \hat{\boldsymbol{\theta}}) \mid \hat{\mathbf{x}}, \hat{\boldsymbol{\theta}} \in S^2 \}$  which is exactly the closed ball of radius  $2k$  centered on the origin in  $\mathbb{R}^3$ . Since  $1 - n(\mathbf{y})$  is of compact support, its Fourier transform is analytic. So knowledge of the Fourier transform on any open ball uniquely determines it.

## References

- Andreas Kirsch, **An Introduction to the Mathematical Theory of Inverse Problems**, Springer, 1996.