Inverse Scattering

Suppose that we are interested in a system in which sound waves, for example, scatter off of some obstacle. Let \( p(x, t) \) be the pressure at position \( x \) and time \( t \). In (a somewhat idealized) free space, \( p \) obeys the wave equation
\[
\frac{\partial^2 p}{\partial t^2} = c^2 \Delta p,
\]
where \( c \) is the speed of sound. We shall assume that in most of the world, \( c \) takes a constant value \( c_0 \). But we introduce an obstacle by allowing \( c \) to depend on position in some compact region. We further allow for some absorption in that region. Then \( p \) obeys
\[
\frac{\partial^2 p}{\partial t^2} + \gamma(x) \frac{\partial p}{\partial t} = c(x)^2 \Delta p,
\]
where \( \gamma(x) \) is the damping coefficient of the medium at \( x \). For solutions of fixed (temporal) frequency, \( p(x, t) = \text{Re} \left[ u(x) e^{-i\omega t} \right] \) with
\[
\Delta u + \frac{\omega^2}{c(x)^2} \left[ 1 + i \frac{\gamma(x)}{\omega} \right] u = 0
\]
Outside of some compact region
\[
\frac{\omega^2}{c(x)^2} \left[ 1 + i \frac{\gamma(x)}{\omega} \right] = \frac{\omega^2}{c_0^2} = k^2 \quad \text{where} \quad k = \frac{\omega}{c_0} > 0
\]
If we define the index of refraction by
\[
n(x) = \frac{c_0^2}{c(x)^2} \left[ 1 + i \frac{\gamma(x)}{\omega} \right]
\]
then
\[
\Delta u + k^2 n(x) u = 0 \quad (1)
\]
with \( n(x) = 1 \) outside of some compact region. We first consider two special cases.

Example 1 (Free Space) In the absence of any obstacle \( \Delta u + k^2 u = 0 \) on all of \( \mathbb{R}^3 \). Then we can solve just by Fourier transforming. The general solution is a mixture of solutions of the form \( u = e^{i k \hat{\theta} \cdot x} \) where \( \hat{\theta} \) is a unit vector. This represents a plane wave coming in from infinity in direction \( \hat{\theta} \).

Example 2 (Point Source) If we have free space everywhere except at the origin and we have a unit point source at the origin, then
\[
\Delta u + k^2 u = \delta(x)
\]
Except at the origin, where there is a singularity, we still have \( \Delta u + k^2 u = 0 \). The point source generates expanding spherical waves. So \( u \) should be a function of \( r = |x| \) only and obey

\[
u''(r) + \frac{2}{r}u'(r) + k^2 u(r) = 0
\]

This is easily solved by changing variables to \( v(r) = ru(r) \), which obeys

\[
v''(r) + k^2 v(r) = 0
\]

So \( v(r) = \alpha \sin(kr) + \beta \cos(kr) \) and \( u(r) = \alpha \frac{\sin(kr)}{r} + \beta \frac{\cos(kr)}{r} \). To be an outgoing (rather than incoming) wave \( u(r) = \alpha' e^{ikr} \). (Note that \( e^{ikr}e^{-i\omega t} \) is constant on \( r = \frac{\omega}{k} t \), which is a sphere that is expanding with speed \( c_0 \).) To give the Dirac delta function on the right hand side of \( \Delta u + k^2 u = \delta(x) \) coefficient one, we need \( u(x) = -\frac{e^{ik|x|}}{4\pi|x|} \). (See, for example, the notes on Poisson’s equation.)

Now let’s return to the general case. We want to think of a physical situation in which we send a plane wave \( u^i(x) = e^{ik\hat{\theta} \cdot \hat{x}} \) in from infinity. This plane wave shakes up the obstacle which then emits a bunch of expanding spherical waves \( e^{ik|x-y|/|x-y|} \) emanating from various points \( y \) in the obstacle. So the full solution is of the form

\[
u(x) = u^i(x) + u^s(x)
\]

where the scattered wave, \( u^s \), obeys the “radiation condition”

\[
\frac{\partial}{\partial r} u^s(x) -iku^s(x) = O\left(\frac{1}{|x|^2}\right) \quad \text{as} \quad |x| \to \infty
\]

This condition is chosen to allow outgoing waves \( e^{ik|x-y|/|x-y|} \) but not incoming waves \( e^{-ik|x-y|/|x-y|} \).

Define

\[
\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}
\]

Since \( \delta(x - y) \) is the kernel of the identity operator,

\[
(\Delta_x + k^2)\Phi(x, y) = -\delta(x - y)
\]

says, roughly, that \( u(x) \mapsto -\int \Phi(x, y)u(y) \, dy \) is the inverse of the map \( u(x) \mapsto (\Delta + k^2)u(x) \) for functions that obey the radiation condition. We can exploit this to convert (1), (2) into an equivalent integral equation

\[
\Delta u + k^2 n(x)u = 0 \quad \implies \quad \Delta u + k^2 u = k^2 (1 - n(x))u
\]

\[
\implies \quad \Delta u^s + k^2 u^s = k^2 (1 - n(x))u
\]
since \( \Delta u^i + k^2 u^i = 0 \). As \( u^s \) obeys the radiation condition

\[
\begin{align*}
    u^s(x) &= -k^2 \int \Phi(x, y)(1 - n(y))u(y) \, dy
\end{align*}
\]

so that

\[
\begin{align*}
    u(x) &= u^i(x) - k^2 \int (1 - n(y))\Phi(x, y)u(y) \, dy
\end{align*}
\]

This is called the Lippmann–Schwinger equation. Observe that it is of the form \( u = u^i - Fu \) or \((\mathbb{I} - F)u = u^i\) where \( F \) is the linear operator \( u(x) \mapsto k^2 \int \Phi(x, y)(1 - n(y))u(y) \, dy \). This operator is compact (if you impose the appropriate norms) and so behaves much like a finite dimensional matrix. If \( F \) has operator norm smaller than one, which is the case if \( k^2(1 - n) \) is small enough, then \( \mathbb{I} - F \) is trivially invertible and the equation \( (\mathbb{I} - F)u = u^i \) has a unique solution. Even if \( F \) has operator norm larger than or equal to one, \( (\mathbb{I} - F)u = u^i \) fails to have a unique solution only if \( F \) has eigenvalue one. One can show that this is impossible in the present setting. Thus, one can prove

**Theorem.** If \( n \in C^2(\mathbb{R}^3) \), \( n(x) - 1 \) has compact support and \( \text{Re} \, n(x), \text{Im} \, n(x) \geq 0 \), then (1), (2) has a unique solution.

For large \( |x| \), \( \Phi \) has the asymptotic behaviour

\[
\Phi(x, y) = \frac{e^{ik|x|}}{4\pi|x|} e^{-ik\hat{x} \cdot y} + O\left(\frac{1}{|x|^2}\right)
\]

so that, when the incoming plane wave is moving in direction \( \hat{\theta} \),

\[
\begin{align*}
    u(x; \hat{\theta}) &= u^i(x; \hat{\theta}) + \frac{e^{ik|x|}}{4\pi|x|} u_\infty(\hat{x}; \hat{\theta}) + O\left(\frac{1}{|x|^2}\right)
\end{align*}
\]

where

\[
\begin{align*}
    u_\infty(\hat{x}; \hat{\theta}) &= -k^2 \int e^{-ik\hat{x} \cdot y} (1 - n(y))u(y; \hat{\theta}) \, dy
\end{align*}
\]

If we are observing the scattered wave from vantage points far from the obstacle, we will only be able to measure \( u_\infty(\hat{x}; \hat{\theta}) \). The inverse problem then is

**Question:** Given \( u_\infty(\hat{x}; \hat{\theta}) \), for all \( \hat{x}, \hat{\theta} \in S^2 \), can we determine \( n \)? The short answer is

**Answer:** Yes, because we have the

**Theorem.** If \( n_1, n_2 \in C^2(\mathbb{R}^3) \) with \( n_1 - 1, n_2 - 1 \) of compact support and

\[
\begin{align*}
    u_{1, \infty}(\hat{x}; \hat{\theta}) &= u_{2, \infty}(\hat{x}; \hat{\theta}), \quad \text{for all } \hat{x}, \hat{\theta} \in S^2, \quad \text{then } n_1 = n_2.
\end{align*}
\]

We can get a rough idea why this Theorem is true by looking at the Born approximation. In this approximation \( u^s \) is ignored in the computation of \( u_\infty \) so that

\[
\begin{align*}
    u_\infty(\hat{x}; \hat{\theta}) &\approx -k^2 \int e^{-ik\hat{x} \cdot y} (1 - n(y))u^i(y; \hat{\theta}) \, dy
\end{align*}
\]

\[
= -k^2 \int e^{-ik(\hat{x} - \hat{\theta}) \cdot y} (1 - n(y)) \, dy
\]
If we measure $u_\infty(\hat{x}; \hat{\theta})$, then, in this approximation, we know the Fourier transform of $1 - n(y)$ on the set \( \{ k(\hat{x} - \hat{\theta}) \mid \hat{x}, \hat{\theta} \in S^2 \} \) which is exactly the closed ball of radius $2k$ centered on the origin in $\mathbb{R}^3$. Since $1 - n(y)$ is of compact support, its Fourier transform is analytic. So knowledge of the Fourier transform on any open ball uniquely determines it.

References