

Poisson's Equation

In these notes we shall find a formula for the solution of Poisson's equation

$$\vec{\nabla}^2 \varphi = 4\pi\rho$$

Here ρ is a given (smooth) function and φ is the unknown function. In electrostatics, ρ is the charge density and φ is the electric potential. The main step in finding this formula will be to consider an

arbitrary (smooth) function φ and an
arbitrary (smooth) region V in \mathbb{R}^3 and an
arbitrary point $\vec{\mathbf{r}}_0$ in the interior of V

and to find a formula which expresses $\varphi(\vec{\mathbf{r}}_0)$ in terms of

$\vec{\nabla}^2 \varphi(\vec{\mathbf{r}})$, with $\vec{\mathbf{r}}$ running over V and
 $\vec{\nabla} \varphi(\vec{\mathbf{r}})$ and $\varphi(\vec{\mathbf{r}})$, with $\vec{\mathbf{r}}$ running only over ∂V .

This formula is

$$\varphi(\vec{\mathbf{r}}_0) = -\frac{1}{4\pi} \left\{ \iiint_V \frac{\vec{\nabla}^2 \varphi(\vec{\mathbf{r}})}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|} d^3 \vec{\mathbf{r}} - \iint_{\partial V} \varphi(\vec{\mathbf{r}}) \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}_0}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|^3} \cdot \hat{\mathbf{n}} dS - \iint_{\partial V} \frac{\vec{\nabla} \varphi(\vec{\mathbf{r}})}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|} \cdot \hat{\mathbf{n}} dS \right\} \quad (V)$$

When we take the limit as V expands to fill all of \mathbb{R}^3 (assuming that φ and $\vec{\nabla} \varphi$ go to zero sufficiently quickly at ∞), we will end up with the formula

$$\varphi(\vec{\mathbf{r}}_0) = -\frac{1}{4\pi} \iiint_V \frac{\vec{\nabla}^2 \varphi(\vec{\mathbf{r}})}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|} d^3 \vec{\mathbf{r}}$$

that expresses φ evaluated at an arbitrary point, $\vec{\mathbf{r}}_0$, of \mathbb{R}^3 in terms of $\vec{\nabla}^2 \varphi(\vec{\mathbf{r}})$, with $\vec{\mathbf{r}}$ running over \mathbb{R}^3 , which is exactly what we want.

Let

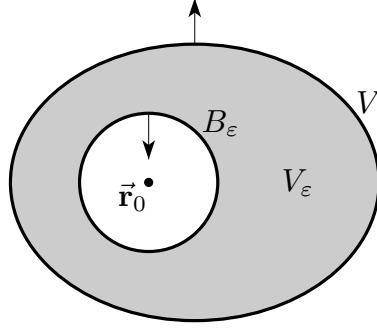
$$\begin{aligned} \vec{\mathbf{r}} &= x \hat{\mathbf{i}} + y \hat{\mathbf{j}} + z \hat{\mathbf{k}} \\ \vec{\mathbf{r}}_0 &= x_0 \hat{\mathbf{i}} + y_0 \hat{\mathbf{j}} + z_0 \hat{\mathbf{k}} \end{aligned}$$

We shall exploit several properties of the function $\frac{1}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|}$. The first two properties are

$$\begin{aligned} \vec{\nabla} \frac{1}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|} &= -\frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}_0}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|^3} \\ \vec{\nabla}^2 \frac{1}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|} &= -\vec{\nabla} \cdot \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}_0}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|^3} = 0 \end{aligned}$$

and are valid for all $\vec{\mathbf{r}} \neq \vec{\mathbf{r}}_0$. Verifying these properties are simple two line computations. The other property of $\frac{1}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|}$ that we shall use is the following. Let B_ε be the sphere of radius ε centered on $\vec{\mathbf{r}}_0$. Then, for any continuous function $\psi(\vec{\mathbf{r}})$,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \iint_{B_\varepsilon} \frac{\psi(\vec{\mathbf{r}})}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|^p} dS &= \lim_{\varepsilon \rightarrow 0^+} \iint_{B_\varepsilon} \frac{\psi(\vec{\mathbf{r}}_0)}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0\|^p} dS = \lim_{\varepsilon \rightarrow 0^+} \frac{\psi(\vec{\mathbf{r}}_0)}{\varepsilon^p} \iint_{B_\varepsilon} dS = \lim_{\varepsilon \rightarrow 0^+} \frac{\psi(\vec{\mathbf{r}}_0)}{\varepsilon^p} 4\pi\varepsilon^2 \\ &= \begin{cases} 4\pi\psi(\vec{\mathbf{r}}_0) & \text{if } p = 2 \\ 0 & \text{if } p < 2 \end{cases} \quad (B) \end{aligned}$$



Here is the derivation of (V). Let V_ϵ be the part of V outside of B_ϵ . Note that the boundary ∂V_ϵ of V_ϵ consists of two parts — the boundary ∂V of V and the sphere B_ϵ — and that the unit outward normal to ∂V_ϵ on B_ϵ is $-\frac{\vec{r}-\vec{r}_0}{\|\vec{r}-\vec{r}_0\|}$. By the divergence theorem

$$\begin{aligned} \iiint_{V_\epsilon} \vec{\nabla} \cdot \left(\frac{1}{\|\vec{r}-\vec{r}_0\|} \vec{\nabla} \varphi - \varphi \vec{\nabla} \frac{1}{\|\vec{r}-\vec{r}_0\|} \right) dV &= \iint_{\partial V} \left(\frac{1}{\|\vec{r}-\vec{r}_0\|} \vec{\nabla} \varphi - \varphi \vec{\nabla} \frac{1}{\|\vec{r}-\vec{r}_0\|} \right) \cdot \hat{\mathbf{n}} dS \\ &+ \iint_{B_\epsilon} \left(\frac{1}{\|\vec{r}-\vec{r}_0\|} \vec{\nabla} \varphi - \varphi \vec{\nabla} \frac{1}{\|\vec{r}-\vec{r}_0\|} \right) \cdot \left(-\frac{\vec{r}-\vec{r}_0}{\|\vec{r}-\vec{r}_0\|} \right) dS \end{aligned} \quad (\text{M})$$

Subbing in $\vec{\nabla} \frac{1}{\|\vec{r}-\vec{r}_0\|} = -\frac{\vec{r}-\vec{r}_0}{\|\vec{r}-\vec{r}_0\|^3}$ and applying (B)

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \iint_{B_\epsilon} \left(\frac{1}{\|\vec{r}-\vec{r}_0\|} \vec{\nabla} \varphi - \varphi \vec{\nabla} \frac{1}{\|\vec{r}-\vec{r}_0\|} \right) \cdot \left(-\frac{\vec{r}-\vec{r}_0}{\|\vec{r}-\vec{r}_0\|} \right) dS &= -\lim_{\epsilon \rightarrow 0^+} \iint_{B_\epsilon} (\vec{\nabla} \varphi \cdot (\vec{r}-\vec{r}_0) + \varphi) \frac{1}{\|\vec{r}-\vec{r}_0\|^2} dS \\ &= -4\pi \left[\vec{\nabla} \varphi \cdot (\vec{r}-\vec{r}_0) + \varphi \right]_{\vec{r}=\vec{r}_0} \\ &= -4\pi \varphi(\vec{r}_0) \end{aligned} \quad (\text{R})$$

Applying $\vec{\nabla} \cdot (f\vec{F}) = \vec{\nabla} f \cdot \vec{F} + f\vec{\nabla} \cdot \vec{F}$, twice, we see that the integrand of the left hand side is

$$\begin{aligned} \vec{\nabla} \cdot \left(\frac{1}{\|\vec{r}-\vec{r}_0\|} \vec{\nabla} \varphi - \varphi \vec{\nabla} \frac{1}{\|\vec{r}-\vec{r}_0\|} \right) &= \vec{\nabla} \frac{1}{\|\vec{r}-\vec{r}_0\|} \cdot \vec{\nabla} \varphi + \frac{1}{\|\vec{r}-\vec{r}_0\|} \vec{\nabla}^2 \varphi - \vec{\nabla} \varphi \cdot \vec{\nabla} \frac{1}{\|\vec{r}-\vec{r}_0\|} - \varphi \vec{\nabla}^2 \frac{1}{\|\vec{r}-\vec{r}_0\|} \\ &= \frac{1}{\|\vec{r}-\vec{r}_0\|} \vec{\nabla}^2 \varphi \end{aligned} \quad (\text{L})$$

since $\vec{\nabla}^2 \frac{1}{\|\vec{r}-\vec{r}_0\|} = 0$ on V_ϵ . So applying $\lim_{\epsilon \rightarrow 0^+}$ to (M) and applying (L) and (R) gives

$$\iiint_V \frac{1}{\|\vec{r}-\vec{r}_0\|} \vec{\nabla}^2 \varphi dV = \iint_{\partial V} \left(\frac{1}{\|\vec{r}-\vec{r}_0\|} \vec{\nabla} \varphi - \varphi \vec{\nabla} \frac{1}{\|\vec{r}-\vec{r}_0\|} \right) \cdot \hat{\mathbf{n}} dS - 4\pi \varphi(\vec{r}_0)$$

which is exactly equation (V).