Consider a wire $0 \leq x \leq \ell$ with voltage $u(x)$ at $x$. By Ohm’s law

$$u(x + dx) - u(x) = -I\rho(x)dx$$

where $I$ is the current flowing through the wire and $\rho(x)dx$ is the resistance between $x$ and $x + dx$. The resistance density $\rho(x)$ is called the resistivity. Dividing across by $dx$ and taking the limit $dx \to 0$

$$u'(x) = -I\rho(x)$$

Assuming that charge is not allowed to accumulate inside the wire, $I$ is a constant and we may eliminate it from the equation just by dividing $\rho(x)$ across and differentiating. If $\gamma(x) = \frac{1}{\rho(x)}$ is the conductivity

$$\gamma(x)u'(x) = -I \implies \left(\gamma(x)u'(x)\right)' = 0 \quad (\star)$$

Now suppose that we may only measure the voltages and currents at the ends of the wire. That is, we may only measure $u(0), u(\ell), \gamma(0)u'(0)$ and $\gamma(\ell)u'(\ell)$. By $(\star)$, $\gamma(x)u'(x)$ is a constant and so takes the value $\gamma(0)u'(0)$ everywhere. Thus

$$u'(x) = \gamma(0)u'(0)\frac{1}{\gamma(x)} \implies u(\ell) - u(0) = \gamma(0)u'(0)\int_0^\ell \frac{dx}{\gamma(x)}$$

The only property of the wire that you can determine by measurements at the ends of the wire is the total resistance $\int_0^\ell \frac{dx}{\gamma(x)}$.

In $\mathbb{R}^n$, $n \geq 2$, the current $i(x)$ is a vector and Ohm’s Law is

$$i(x) = -\gamma(x)\nabla u(x)$$

Assuming that charge is not allowed to accumulate, the net rate of charge flow across the boundary $\partial V$ of any region $V$ must vanish, so that

$$\int_{\partial V} i(x) \cdot \hat{n}dS = 0$$

By the divergence theorem

$$\nabla \cdot i(x) = 0 \implies \nabla \cdot (\gamma(x)\nabla u(x)) = 0$$

Suppose now that we have a conductor filling a region $\Omega$ and that we apply a voltage $f$ on the boundary $\partial \Omega$ of $\Omega$ and measure the current that then flows out of the region. By measuring the current exiting various parts of $\partial \Omega$, we are measuring the current flux on $\partial \Omega$,
which determines $\gamma(x) \frac{\partial u}{\partial \nu}(x)$ on $\partial \Omega$, where $\frac{\partial u}{\partial \nu}$ is the normal derivative. For a given $\gamma$ and $f$, the boundary value problem

$$\nabla \cdot (\gamma(x) \nabla u(x)) = 0 \quad \text{in } \Omega \quad \text{and} \quad u = f \quad \text{on } \partial \Omega$$

determines $u$ on $\Omega$ and hence $k(x) = \gamma(x) \frac{\partial u}{\partial \nu}(x) \upharpoonright \partial \Omega$. Let $\Lambda_\gamma(f)$ be the $k$ that results from a given $\gamma$ and $f$. Clearly $\Lambda_\gamma(f)$ depends linearly on $f$. The map

$$\Lambda_\gamma : C^\infty(\partial \Omega) \to C^\infty(\partial \Omega)$$

is called the Dirichlet to Neumann Map. Because $\Lambda_\gamma$ is a linear map on $C^\infty(\partial \Omega)$, it has a distributional kernel

$$\Lambda_\gamma(f) = \int_{\partial \Omega} \lambda_\gamma(x, y) f(y) \, dS(y)$$

where $dS$ is the surface measure on $\partial \Omega$. If we measure the current $k$ that results from all applied surface voltages $f$, we know $\lambda_\gamma(x, y)$ for all $x, y \in \partial \Omega$. This is a function of $2(n - 1)$ variables. The conductivity $\gamma(x)$ is a function of $n$ variables. So for $n = 1$, $\gamma(x)$ is a function of more variables than $\lambda_\gamma(x, y)$. We have already seen that, for $n = 1$, $\lambda_\gamma(x, y)$ cannot possibly determine $\gamma(x)$. For $n = 2$ ($n > 2$), $\gamma(x)$ is a function of the same number of variables as (fewer variables than) $\lambda_\gamma(x, y)$.

In general, $\gamma(x)$ is a positive definite, symmetric, $n \times n$ matrix. If $\gamma(x)$ is scalar (that is, a multiple of the identity matrix), the medium is called isotropic. Otherwise it is called anisotropic. In Uhlmann’s notes, it is proven that, for $n \geq 2$, $\Lambda_\gamma$ does indeed determine an isotropic conductivity. However, it cannot possibly determine anisotropic conductivities for the following obvious reason. Let $\Psi : \tilde{\Omega} \to \Omega$ be a diffeomorphism with $\Psi \upharpoonright \partial \Omega$ being the identity map. Given any $u, \gamma$, set

$$\tilde{\gamma} = \frac{1}{|\det(D\Psi)|} (D\Psi)^t \gamma (D\Psi) \circ \Psi^{-1} \quad \tilde{u} = u \circ \Psi^{-1}$$

where $D\Psi$ is the Jacobian (matrix of first partial derivatives) of $\Psi$. Then

$$\nabla \cdot (\gamma(x) \nabla u(x)) = 0 \quad \text{in } \Omega \quad \text{and} \quad u = f \quad \text{on } \partial \Omega \quad \implies \quad \nabla \cdot (\tilde{\gamma}(x) \nabla \tilde{u}(x)) = 0 \quad \text{in } \Omega \quad \tilde{u} = f \quad \text{on } \partial \Omega$$

Thus $\Lambda_\gamma = \Lambda_{\tilde{\gamma}}$. In Uhlmann’s notes, it is proven that, for $n = 2$, $\Lambda_\gamma$ determines anisotropic conductivities up to diffeomorphisms like this. He conjectures that this is also true for $n > 2$.

**Example.** Here is a carefully rigged example in which an isotropic conductivity is computed from a Dirichlet to Neumann map. The region $\Omega = [0, 1]^2$ is square. We assume that we know

1. $\nabla \cdot (\gamma(x_1) \nabla u(x)) = 0$ in $\Omega$
2. $u(0, x_2) = u(1, x_2) = \sin \pi x_2$ for all $0 \leq x_2 \leq 1$
3. $u(x_1, 0) = u(x_1, 1) = 0$ for all $0 \leq x_1 \leq 1$

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Note that we are assuming that the conductivity is isotropic and also is a function of \( x_1 \) only. Motivated by (1) and (2), we look for a solution of the form \( u(x_1, x_2) = a(x_1) \sin(\pi x_2) \). Condition (3) is satisfied for all \( a(x_1) \). Condition (2) is satisfied if and only if \( a(0) = a(1) = 1 \). Condition (1) is satisfied if and only if

\[
0 = \nabla \cdot (\gamma(x_1)a'(x_1) \sin \pi x_2, \gamma(x_1)a(x_1) \pi \cos \pi x_2)
\]

\[
= \sin \pi x_2 \left[ (\gamma(x_1)a'(x_1))' - \pi^2 \gamma(x_1)a(x_1) \right]
\]

which is the case if and only if

\[
(4) \quad (\gamma(x_1)a'(x_1))' - \pi^2 \gamma(x_1)a(x_1) = 0 \quad \text{for all } 0 \leq x_1 \leq 1
\]

We imagine that we have measured

\[
k(x_1) = \gamma(x_1) \frac{\partial u}{\partial x_2} \bigg|_{x_2=0} = \gamma(x_1) \pi a(x_1) \cos \pi x_2 \bigg|_{x_2=0} = \pi \gamma(x_1)a(x_1)
\]

and that we wish to determine \( \gamma(x_1) \). We can do so by subbing \( \gamma(x_1) = \frac{k(x_1)}{\pi a(x_1)} \) into (4) and solving for \( a \).

\[
(k(x_1) \frac{a'(x_1)}{a(x_1)})' = \pi^2 k(x_1) \implies \frac{d}{dx_1} \left[ k(x_1) \frac{d}{dx_1} \ln a(x_1) \right] = \pi^2 k(x_1)
\]

\[
\implies k(x_1) \frac{d}{dx_1} \ln a(x_1) = \pi^2 \int_0^{x_1} k(t) \, dt - \pi^2 \mathcal{C}
\]

\[
\implies \ln a(x_1) = \pi^2 \int_0^{x_1} \frac{1}{k(s)} \left[ \int_0^s k(t) \, dt - \mathcal{C} \right] \, ds + D
\]

To satisfy the boundary condition \( a(0) = 1 \), we need \( D = 0 \) and to satisfy \( a(1) = 1 \), we need

\[
C = \left[ \int_0^1 \frac{ds}{k(s)} \right]^{-1} \left[ \int_0^1 \frac{ds}{k(s)} \int_0^s k(t) \, dt \right]
\]

This determines\(^{(1)}\) \( a \) and hence \( \gamma = \frac{k}{\pi a} \).

References


\(^{(1)}\) If you are worried about dividing by \( k \) in the integrals, you shouldn’t be. We know that \( 0 \leq u \leq 1 \) on \( \partial \Omega \). By the maximum principle, this implies that \( 0 < u < 1 \) in the interior of \( \Omega \). This in turn forces \( \frac{\partial u}{\partial x_2} \geq 0 \) when \( x_2 = 0 \). In fact, by the strong maximum principle, \( \frac{\partial u}{\partial x_2} > 0 \) for \( x_2 = 0 \), which ensures that \( k(x_1) > 0 \) for all \( 0 \leq x_1 \leq 1 \).