

Dirichlet to Neumann Problems

Consider a wire $0 \leq x \leq \ell$ with voltage $u(x)$ at x . By Ohm's law

$$u(x + dx) - u(x) = -I\rho(x)dx$$

where I is the current flowing through the wire and $\rho(x)dx$ is the resistance between x and $x + dx$. The resistance density $\rho(x)$ is called the resistivity. Dividing across by dx and taking the limit $dx \rightarrow 0$

$$u'(x) = -I\rho(x)$$

Assuming that charge is not allowed to accumulate inside the wire, I is a constant and we may eliminate it from the equation just by dividing $\rho(x)$ across and differentiating. If $\gamma(x) = \frac{1}{\rho(x)}$ is the conductivity

$$\gamma(x)u'(x) = -I \implies (\gamma(x)u'(x))' = 0 \quad (*)$$

Now suppose that we may only measure the voltages and currents at the ends of the wire. That is, we may only measure $u(0), u(\ell), \gamma(0)u'(0)$ and $\gamma(\ell)u'(\ell)$. By (*), $\gamma(x)u'(x)$ is a constant and so takes the value $\gamma(0)u'(0)$ everywhere. Thus

$$u'(x) = \gamma(0)u'(0)\frac{1}{\gamma(x)} \implies u(\ell) - u(0) = \gamma(0)u'(0) \int_0^\ell \frac{dx}{\gamma(x)}$$

The only property of the wire that you can determine by measurements at the ends of the wire is the total resistance $\int_0^\ell \frac{dx}{\gamma(x)}$.

In \mathbb{R}^n , $n \geq 2$, the current $\mathbf{i}(\mathbf{x})$ is a vector and Ohm's Law is

$$\mathbf{i}(\mathbf{x}) = -\gamma(\mathbf{x})\nabla u(\mathbf{x})$$

Assuming that charge is not allowed to accumulate, the net rate of charge flow across the boundary ∂V of any region V must vanish, so that

$$\int_{\partial V} \mathbf{i}(\mathbf{x}) \cdot \hat{\mathbf{n}} dS = 0$$

By the divergence theorem

$$\nabla \cdot \mathbf{i}(\mathbf{x}) = 0 \implies \nabla \cdot (\gamma(\mathbf{x})\nabla u(\mathbf{x})) = 0$$

Suppose now that we have a conductor filling a region Ω and that we apply a voltage f on the boundary $\partial\Omega$ of Ω and measure the current that then flows out of the region. By measuring the current exiting various parts of $\partial\Omega$, we are measuring the current flux on $\partial\Omega$,

which determines $\gamma(\mathbf{x})\frac{\partial u}{\partial \nu}(\mathbf{x})$ on $\partial\Omega$, where $\frac{\partial u}{\partial \nu}$ is the normal derivative. For a given γ and f , the boundary value problem

$$\nabla \cdot (\gamma(\mathbf{x})\nabla u(\mathbf{x})) = 0 \text{ in } \Omega \quad u = f \text{ on } \partial\Omega$$

determines u on Ω and hence $k(\mathbf{x}) = \gamma(\mathbf{x})\frac{\partial u}{\partial \nu}(\mathbf{x}) \upharpoonright \partial\Omega$. Let $\Lambda_\gamma(f)$ be the k that results from a given γ and f . Clearly $\Lambda_\gamma(f)$ depends linearly on f . The map

$$\Lambda_\gamma : C^\infty(\partial\Omega) \rightarrow C^\infty(\partial\Omega)$$

is called the Dirichlet to Neumann Map. Because Λ_γ is a linear map on $C^\infty(\partial\Omega)$, it has a distributional kernel

$$\Lambda_\gamma(f) = \int_{\partial\Omega} \lambda_\gamma(x, y) f(y) dS(y)$$

where dS is the surface measure on $\partial\Omega$. If we measure the current k that results from all applied surface voltages f , we know $\lambda_\gamma(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \partial\Omega$. This is a function of $2(n-1)$ variables. The conductivity $\gamma(\mathbf{x})$ is a function of n variables. So for $n=1$, $\gamma(\mathbf{x})$ is a function of more variables than $\lambda_\gamma(\mathbf{x}, \mathbf{y})$. We have already seen that, for $n=1$, $\lambda_\gamma(\mathbf{x}, \mathbf{y})$ cannot possibly determine $\gamma(\mathbf{x})$. For $n=2$ ($n > 2$), $\gamma(\mathbf{x})$ is a function of the same number of variables as (fewer variables than) $\lambda_\gamma(\mathbf{x}, \mathbf{y})$.

In general, $\gamma(\mathbf{x})$ is a positive definite, symmetric, $n \times n$ matrix. If $\gamma(\mathbf{x})$ is scalar (that is, a multiple of the identity matrix), the medium is called isotropic. Otherwise it is called anisotropic. In Uhlmann's notes, it is proven that, for $n \geq 2$, Λ_γ does indeed determine an isotropic conductivity. However, it cannot possibly determine anisotropic conductivities for the following obvious reason. Let $\Psi : \bar{\Omega} \rightarrow \bar{\Omega}$ be a diffeomorphism with $\Psi \upharpoonright \partial\Omega$ being the identity map. Given any u, γ , set

$$\tilde{\gamma} = \frac{1}{|\det(D\Psi)|} (D\Psi)\gamma(D\Psi)^t \circ \Psi^{-1} \quad \tilde{u} = u \circ \Psi^{-1}$$

where $D\Psi$ is the Jacobian (matrix of first partial derivatives) of Ψ . Then

$$\nabla \cdot (\gamma(\mathbf{x})\nabla u(\mathbf{x})) = 0 \text{ in } \Omega \quad u = f \text{ on } \partial\Omega \quad \implies \quad \nabla \cdot (\tilde{\gamma}(\mathbf{x})\nabla \tilde{u}(\mathbf{x})) = 0 \text{ in } \Omega \quad \tilde{u} = f \text{ on } \partial\Omega$$

Thus $\Lambda_\gamma = \Lambda_{\tilde{\gamma}}$. In Uhlmann's notes, it is proven that, for $n=2$, Λ_γ determines anisotropic conductivities up to diffeomorphisms like this. He conjectures that this is also true for $n > 2$.

Example. Here is a carefully rigged example in which an isotropic conductivity is computed from a Dirichlet to Neumann map. The region $\Omega = [0, 1]^2$ is square. We assume that we know

- (1) $\nabla \cdot (\gamma(x_1)\nabla u(\mathbf{x})) = 0$ in Ω
- (2) $u(0, x_2) = u(1, x_2) = \sin \pi x_2$ for all $0 \leq x_2 \leq 1$
- (3) $u(x_1, 0) = u(x_1, 1) = 0$ for all $0 \leq x_1 \leq 1$

Note that we are assuming that the conductivity is isotropic and also is a function of x_1 only. Motivated by (1) and (2), we look for a solution of the form $u(x_1, x_2) = a(x_1) \sin(\pi x_2)$. Condition (3) is satisfied for all $a(x_1)$. Condition (2) is satisfied if and only if $a(0) = a(1) = 1$. Condition (1) is satisfied if and only if

$$\begin{aligned} 0 &= \nabla \cdot (\gamma(x_1)a'(x_1) \sin \pi x_2, \gamma(x_1)a(x_1)\pi \cos \pi x_2) \\ &= \sin \pi x_2 [(\gamma(x_1)a'(x_1))' - \pi^2 \gamma(x_1)a(x_1)] \end{aligned}$$

which is the case if and only if

$$(4) \quad (\gamma(x_1)a'(x_1))' - \pi^2 \gamma(x_1)a(x_1) = 0 \quad \text{for all } 0 \leq x_1 \leq 1$$

We imagine that we have measured

$$k(x_1) = \gamma(x_1) \frac{\partial u}{\partial x_2} \Big|_{x_2=0} = \gamma(x_1) \pi a(x_1) \cos \pi x_2 \Big|_{x_2=0} = \pi \gamma(x_1) a(x_1)$$

and that we wish to determine $\gamma(x_1)$. We can do so by subbing $\gamma(x_1) = \frac{k(x_1)}{\pi a(x_1)}$ into (4) and solving for a .

$$\begin{aligned} (k(x_1) \frac{a'(x_1)}{a(x_1)})' = \pi^2 k(x_1) &\implies \frac{d}{dx_1} [k(x_1) \frac{d}{dx_1} \ln a(x_1)] = \pi^2 k(x_1) \\ &\implies k(x_1) \frac{d}{dx_1} \ln a(x_1) = \pi^2 \int_0^{x_1} k(t) dt - \pi^2 C \\ &\implies \ln a(x_1) = \pi^2 \int_0^{x_1} \frac{1}{k(s)} \left[\int_0^s k(t) dt - C \right] ds + D \end{aligned}$$

To satisfy the boundary condition $a(0) = 1$, we need $D = 0$ and to satisfy $a(1) = 1$, we need

$$C = \left[\int_0^1 \frac{ds}{k(s)} \right]^{-1} \left[\int_0^1 \frac{ds}{k(s)} \int_0^s k(t) dt \right]$$

This determines⁽¹⁾ a and hence $\gamma = \frac{k}{\pi a}$.

References

- Gunther Uhlmann, **The Dirichlet to Neumann Map and Inverse Problems**, preprint.

⁽¹⁾ If you are worried about dividing by k in the integrals, you shouldn't be. We know that $0 \leq u \leq 1$ on $\partial\Omega$. By the maximum principle, this implies that $0 < u < 1$ in the interior of Ω . This in turn forces $\frac{\partial u}{\partial x_2} \geq 0$ when $x_2 = 0$. In fact, by the strong maximum principle, $\frac{\partial u}{\partial x_2} > 0$ for $x_2 = 0$, which ensures that $k(x_1) > 0$ for all $0 \leq x_1 \leq 1$.