

# Resumé - holomorphic/harmonic forms on compact Riemann surfaces

**Notation.**

$M =$  a compact Riemann surface of genus  $g$   
with canonical homology basis  $\{a_1, \dots, a_g, b_1, \dots, b_g\}$

$$\aleph_j = \begin{cases} a_j & \text{if } 1 \leq j \leq g \\ b_{j-g} & \text{if } g+1 \leq j \leq 2g \end{cases}$$

$$H = \{ \omega \in L^2(M) \mid \omega \text{ harmonic} \}$$

$$\mathcal{H} = \{ \omega \in L^2(M) \mid \omega \text{ holomorphic} \}$$

**Theorem.**

- a)  $\dim H = 2g$
- b) *There is a unique basis  $\alpha_1, \dots, \alpha_{2g}$  of  $H$  obeying*

$$\int_{\aleph_j} \alpha_k = \delta_{j,k}$$

*Furthermore,  $\alpha_k = \overline{\alpha_k}$  for all  $1 \leq k \leq 2g$  and*

$$\iint_M \alpha_j \wedge \alpha_k = \aleph_j \cdot \aleph_k = \begin{cases} 0 & \text{if } 1 \leq j, k \leq g \\ 0 & \text{if } g+1 \leq j, k \leq 2g \\ \delta_{j,k-g} & \text{if } 1 \leq j, k-g \leq g \\ -\delta_{j-g,k} & \text{if } 1 \leq j-g, k \leq g \end{cases}$$

*for all  $1 \leq j, k \leq 2g$ .*

**Proposition.** *Let  $\theta, \tilde{\theta}$  be a closed 1-forms on  $M$ . Then*

$$\iint_M \theta \wedge \tilde{\theta} = \sum_{j=1}^g \left[ \int_{a_j} \theta \int_{b_j} \tilde{\theta} - \int_{b_j} \theta \int_{a_j} \tilde{\theta} \right]$$

**Corollary.** *If  $\theta \in H$ , then*

$$\|\theta\|^2 = \sum_{j=1}^g \left[ \int_{a_j} \theta \int_{b_j} *\bar{\theta} - \int_{b_j} \theta \int_{a_j} *\bar{\theta} \right]$$

*If  $\theta \in \mathcal{H}$ , then*

$$\|\theta\|^2 = \sum_{j=1}^g i \left[ \int_{a_j} \theta \int_{b_j} \bar{\theta} - \int_{b_j} \theta \int_{a_j} \bar{\theta} \right]$$

**Theorem.**

- a)  $\dim \mathcal{H} = g$
- b) *There is a unique basis  $\omega_1, \dots, \omega_g$  of  $\mathcal{H}$  obeying*

$$\int_{\mathbf{a}_j} \omega_k = \delta_{j,k}$$

*Furthermore,  $\Pi_{j,k} = \int_{b_j} \omega_k$  obeys*

- $\Pi_{j,k} = \Pi_{k,j}$  for all  $1 \leq j, k \leq g$  ( $\Pi$  is symmetric) and
  - $\frac{1}{2i} \sum_{1 \leq j, k \leq g} \bar{\mu}_j (\Pi_{j,k} - \overline{\Pi_{j,k}}) \mu_k \geq 0$  for all  $\vec{\mu} \in \mathbb{C}^g$ , with equality if and only if  $\vec{\mu} = 0$  ( $\text{Im } \Pi$  is positive definite).
- c) *There is a unique  $\theta \in \mathcal{H}$  with prescribed values of  $\int_{a_1} \theta, \dots, \int_{a_g} \theta$ .*
  - c') *There is a unique  $\theta \in \mathcal{H}$  with prescribed values of  $\text{Re} \int_{a_1} \theta, \dots, \text{Re} \int_{a_g} \theta, \text{Re} \int_{b_1} \theta, \dots, \text{Re} \int_{b_g} \theta$ .*