Resumé - closed/exact/holomorphic/harmonic forms

Definition.

a) A 1–form ω is (co–)closed if ω is C^1 and $d(*)\omega = 0$.

b) A 1–form ω is (co–)exact if $\omega = (*)dF$ for some C^2 function F on M.

Proposition. *Let ω be a C^1 1–form.*

a) If ω is (co–)exact, then ω is (co–)closed.

b) ω is (co–)closed if and only if $\int_{\delta D}(*)\omega = 0$ for all 2–chains D.

c) ω is (co–)exact if and only if $\int_c (*)\omega = 0$ for all closed 1–chains c.

Definition.

a) A 0–form (function) F is harmonic if F is C^2 and $\Delta F = 0$.

b) A 1–form ω is harmonic if, for each $x \in M$, there is a neighbourhood U of x and a harmonic function F such that $\omega\big|_U = dF\big|_U$.

c) A 1–form ω is holomorphic if, for each $x \in M$, there is a neighbourhood U of x and a holomorphic function F such that $\omega\big|_U = dF\big|_U$.

Proposition.

A differential form ω is holomorphic

$\iff$ *there is an atlas $\mathcal{A}$ such that for each patch $\{U, \zeta\} \in \mathcal{A}$,*
$\omega\big|_{\{U,\zeta\}} = u dz + v d\bar{z}$ *with $v = 0$ and u holomorphic*

$\iff$ *for all coordinate patches $\{U, \zeta\}$ of M,*
$\omega\big|_{\{U,\zeta\}} = u dz + v d\bar{z}$ *with $v = 0$ and u holomorphic*

$\iff$ *ω is closed and $*\omega = -i\omega$*

$\iff$ *$\omega = \alpha + i * \alpha$ for some harmonic differential α*

Proposition.

A differential form α is harmonic

$\iff$ *α is closed and co–closed.*

$\iff$ *$\alpha = \omega_1 + \bar{\omega}_2$ with ω_1, ω_2 holomorphic*

Definition.

$$E = \left\{ \, df \;\middle|\; f \in C_0^\infty(M) \, \right\}^-$$
$$E^* = \left\{ \, *df \;\middle|\; f \in C_0^\infty(M) \, \right\}^-$$
$$H = \left\{ \, \omega \in L^2(M) \;\middle|\; \omega \text{ harmonic} \, \right\}$$

Theorem. *Let $\alpha \in L^2(M) \cap C^1$.*

a) $L^2(M) = E \oplus E^* \oplus H$

b) α *closed* $\iff \alpha \in (E^*)^\perp = E \oplus H$

c) α *co–closed* $\iff \alpha \in E^\perp = E^* \oplus H$

d) α *(co–)exact* $\Longleftarrow \alpha \in E^{(*)}$.

If M is compact, α (co–)exact $\iff \alpha \in E^{(*)}$

Proposition. *Let c be a closed 1–chain in M. There exists a closed, C_0^∞, real 1–form η_c such that*

$$\int_c \alpha = (\alpha, *\eta_c)$$

for all closed $\alpha \in L^2(M) \cap C^1$.

Corollary. *Let $\alpha \in L^2(M) \cap C^1$. Then α is exact (co–exact) if and only if $(\alpha, \beta) = 0$ for all co–closed (closed) C_0^∞ 1–forms β.*