

Projective Curves

The n dimensional complex projective space is the set of all equivalence classes

$$\mathbb{C}\mathbb{P}^n = \{ [z_1, \dots, z_{n+1}] \mid (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\} \}$$

under the equivalence relation

$$(z_1, \dots, z_{n+1}) \sim (z'_1, \dots, z'_{n+1}) \iff \exists z \in \mathbb{C} \setminus \{0\} \text{ such that } (z'_1, \dots, z'_{n+1}) = z(z_1, \dots, z_{n+1})$$

We can think of $\mathbb{C}\mathbb{P}^n$ as \mathbb{C}^n , which we identify with $\{ [z_1, \dots, z_n, 1] \mid (z_1, \dots, z_n) \in \mathbb{C}^n \}$, with some points at infinity tacked on. Since $[z_1, \dots, z_n, z] = [\frac{z_1}{z}, \dots, \frac{z_n}{z}, 1]$ for all $z \neq 0$, the set of points in $\mathbb{C}\mathbb{P}^n$ which we have not identified with points in \mathbb{C}^n is $\{ [z_1, \dots, z_n, 0] \mid (z_1, \dots, z_n) \in \mathbb{C}^n \setminus \{(0, \dots, 0)\} \}$, which is just $\mathbb{C}\mathbb{P}^{n-1}$. This is the set of points at infinity. Each complex line in \mathbb{C}^n that passes through the origin is of the form $\{ z(z_1, \dots, z_n) \mid z \in \mathbb{C} \}$ for some $(z_1, \dots, z_n) \in \mathbb{C}^n \setminus \{(0, \dots, 0)\}$. (It has real dimension two, but complex geometers still call it a line because it has complex dimension one.) There is one point at infinity $\mathbb{C}\mathbb{P}^n$ for each complex line in \mathbb{C}^n . Since

$$[z_1, \dots, z_n, 0] = \lim_{z \rightarrow 0} [z_1, \dots, z_n, z] = \lim_{z \rightarrow 0} [\frac{z_1}{z}, \dots, \frac{z_n}{z}, 1]$$

and $[\frac{z_1}{z}, \dots, \frac{z_n}{z}, 1]$ is identified with the point $\frac{1}{z}(z_1, \dots, z_n) \in \mathbb{C}^n$, you can get to the point $[z_1, \dots, z_n, 0]$ at infinity in $\mathbb{C}\mathbb{P}^n$ by “going to infinity” along the complex line in \mathbb{C}^n that is associated with $[z_1, \dots, z_n, 0]$.

In general, a function $F(z_1, \dots, z_{n+1})$ on \mathbb{C}^{n+1} does not make sense as a function on $\mathbb{C}\mathbb{P}^n$ because F can take different values at equivalent points $(z_1, \dots, z_{n+1}) \sim (z'_1, \dots, z'_{n+1})$. But if F is a homogeneous polynomial of degree d , then $F(zz_1, \dots, zz_{n+1}) = z^d F(z_1, \dots, z_{n+1})$ so that at least

$$F(z_1, \dots, z_{n+1}) = 0 \iff F(z'_1, \dots, z'_{n+1}) = 0 \text{ for all } (z'_1, \dots, z'_{n+1}) \sim (z_1, \dots, z_{n+1})$$

Thus the zero set

$$M_F = \{ [z_1, \dots, z_{n+1}] \in \mathbb{C}\mathbb{P}^n \mid F(z_1, \dots, z_{n+1}) = 0 \}$$

is a well defined subset of $\mathbb{C}\mathbb{P}^n$. If F is nonsingular, meaning that there are no solutions to the system of equations

$$F = \frac{\partial F}{\partial z_1} = \dots = \frac{\partial F}{\partial z_{n+1}} = 0$$

then M_F defines a smooth $n - 1$ (complex) dimensional manifold in $\mathbb{C}\mathbb{P}^n$. If $n = 2$ then M_F is a Riemann surface. (It turns out that connectedness is automatic in this case. Disconnectedness in \mathbb{C}^2 gives a singularity at infinity in $\mathbb{P}\mathbb{C}^2$. For example: $f(z_1, z_2) = z_1(z_1 - 1)$, $F(z_1, z_2, z_3) = z_1(z_1 - z_3)$.) If $n > 2$, we can also get Riemann surfaces by taking the intersection $M_{F_1} \cap \dots \cap M_{F_{n-1}}$ of $n - 1$ such surfaces. The intersection is smooth if the $(n - 1) \times (n + 1)$ matrix $(\frac{\partial F_i}{\partial z_j})$ of partial derivatives has maximal rank $n - 1$. Again, it turns out that smoothness implies connectedness.

If f is any polynomial on \mathbb{C}^n , we can always find a homogeneous polynomial F on \mathbb{C}^{n+1} with the same degree as f , such that the zero set of f in \mathbb{C}^n and the part of M_F with $z_{n+1} = 1$ (i.e. excluding the part at infinity) coincide under the identification we discussed above. For example, if $f(x, y) = y^2 - x^3 + x$ (whose zero set is the elliptic curve we saw in class), then $F(x, y, z) = y^2z - x^3 + xz^2$. The advantage of M_F is that it is always compact, since $\mathbb{C}\mathbb{P}^n$ is compact.