Math 602 Problem Set III

Due Friday March 1, 2002

1) Define, for any differentiable function $f(x,y)$,

$$f_z(x,y) = \tfrac{1}{2}\big(\tfrac{\partial f}{\partial x}(x,y) - i\tfrac{\partial f}{\partial y}(x,y)\big) \qquad f_{\bar z}(x,y) = \tfrac{1}{2}\big(\tfrac{\partial f}{\partial x}(x,y) + i\tfrac{\partial f}{\partial y}(x,y)\big)$$

Show that

a) $f_z(x,y) = \overline{f_{\bar z}(x,y)}$

b) If $f(x,y) = F(x+iy)$ with F analytic, then $f_z(x,y) = F'(x+iy)$ and $f_{\bar z}(x,y) = 0$.

c) If $f(x,y) = F(x-iy)$ with F analytic, then $f_z(x,y) = 0$ and $f_{\bar z}(x,y) = F'(x-iy)$.

2) Show that, if ω is a 1–form, then $*\omega$ is a well–defined 1–form. That is, show that $*\omega$ transforms correctly under holomorphic changes of coordinates.

3) Show that, if F is a 0–form, then ΔF is a well–defined 2–form. That is, show that ΔF transforms correctly under holomorphic changes of coordinates. Also show that

$$\Delta F = d * dF = -2i\bar\partial\partial F$$

4) Define, for any 1–form ω and any coordinate patch $\{U,\zeta\}$

$$\bar\omega\big|_{\{U,\zeta\}} = \overline{f(x,y)}dx + \overline{g(x,y)}dy \text{ where } \omega\big|_{\{U,\zeta\}} = f(x,y)dx + g(x,y)dy$$

a) Show that $\bar\omega$ is a well–defined 1–form. That is, show that $\bar\omega$ transforms correctly under changes of coordinates.

b) Show that, if $\omega = u(x,y)\,dz + v(x,y)\,d\bar z$ in some coordinate patch, then, in that patch, $\bar\omega = \overline{v(x,y)}\,dz + \overline{u(x,y)}\,d\bar z$.

c) Show that $\overline{*\omega} = *\bar\omega$.

5) Let ω be a harmonic 1–form on the Riemann surface M. Suppose that ω vanishes on a convergent sequence of distinct points of M. Prove that ω vanishes on M.

6) Let $0 \le a < b \le \infty$ and $D = \big\{\ z \in \mathbb{C}\ \big|\ a < |z| < b\ \big\}$. Let $\mathbb{C}_\infty$ be the extended complex plane and $T = \mathbb{C}/\mathbb{Z}^2$. Let $H(M)$ be the space of all L^2 harmonic 1–forms on the Riemann surface M and let $\mathcal{H}(M)$ be the space of all L^2 holomorphic 1–forms on the Riemann surface M. Find an orthonormal basis for each of $H(D)$, $H(\mathbb{C}_\infty)$, $H(T)$, $\mathcal{H}(D)$, $\mathcal{H}(\mathbb{C}_\infty)$ and $\mathcal{H}(T)$.