

**Math 602 Problem Set I**

Due January 25

1) Let “diffalytic” stand for either “infinitely differentiable” or “analytic”. Let  $\mathcal{A}$  be a diffalytic atlas for a manifold  $M$ . An atlas is diffalytic if the transition functions for each pair of charts in the atlas are diffalytic. Prove that there is a unique maximal diffalytic atlas for  $M$  that contains  $\mathcal{A}$ .

2) Let  $S = \{ (x, y, t) \in \mathbb{R}^3 \mid x^2 + y^2 + t^2 = 1 \}$  and define the map  $p : \mathbb{C}_\infty \rightarrow S$  as follows. Set  $p(\infty) = (0, 0, 1)$ . For each  $z = x + iy \in \mathbb{C}$ , set  $p(z)$  to the point of intersection (other than  $(0, 0, 1)$ ) between  $S$  and the line through  $(0, 0, 1)$  and  $(x, y, 0)$ .

a) Show that, for  $z = x + iy \in \mathbb{C}$ ,

$$p(z) = \left( \frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$$

b) Define  $S_1 = S \setminus \{(0, 0, 1)\}$ ,  $S_2 = S \setminus \{(0, 0, -1)\}$ ,  $U_1 = \mathbb{C}_\infty \setminus \{\infty\}$ ,  $U_2 = \mathbb{C}_\infty \setminus \{0\}$  and the maps  $\zeta_i : U_i \rightarrow \mathbb{C}$ ,  $\sigma_i : S_i \rightarrow \mathbb{C}$  by

$$\zeta_1(z) = z \quad \zeta_2(z) = \frac{1}{z} \quad \sigma_1(x, y, t) = \frac{x+iy}{1-t} \quad \sigma_2(x, y, t) = \frac{x-iy}{1+t}$$

with the convention  $\frac{1}{\infty} = 0$ . Show that  $\sigma_i \circ p \circ \zeta_i^{-1}(z) = z$  for  $i = 1, 2$  and all  $z \in \mathbb{C}$ .

c) Let  $P$  be a plane in  $\mathbb{R}^3$  that intersects  $S$  in more than one point. Show that  $p^{-1}(P \cap S)$  is a circle in  $\mathbb{C}$  (if  $(0, 0, 1) \notin P$ ) or is  $L \cup \{\infty\}$  for some line  $L$  (if  $(0, 0, 1) \in P$ ). For this reason, we usually regard  $L \cup \{\infty\}$  as a circle in  $\mathbb{C}_\infty$ .

3) Let  $f(z) = \cos z$  and let  $\Gamma = \{ (z, w) \in \mathbb{C}^2 \mid w = f(z) \}$  be its graph. Denote by  $\zeta(z, w) = z$  and  $\omega(z, w) = w$  the projections from  $\Gamma$  to the  $z$ - and  $w$ -planes, respectively.

a) Prove that  $f(z) = f(z')$  if and only if  $z - z' \in 2\pi\mathbb{Z}$  or  $z + z' \in 2\pi\mathbb{Z}$ .

b) Sketch, in a single figure, the images under  $f$  of the following sets

$$\begin{aligned} & \{ z \in \mathbb{C} \mid 0 < \operatorname{Re} z \leq \pi, \operatorname{Im} z > 0 \} & \{ z \in \mathbb{C} \mid 0 < \operatorname{Re} z \leq \pi, \operatorname{Im} z < 0 \} \\ & \{ z \in \mathbb{C} \mid \operatorname{Re} z = 0, \operatorname{Im} z > 0 \} & \{ z \in \mathbb{C} \mid \operatorname{Re} z = 0, \operatorname{Im} z < 0 \} \\ & \{ z \in \mathbb{C} \mid \operatorname{Re} z = \pi, \operatorname{Im} z > 0 \} & \{ z \in \mathbb{C} \mid \operatorname{Re} z = \pi, \operatorname{Im} z < 0 \} \\ & \{ z \in \mathbb{C} \mid 0 < \operatorname{Re} z \leq \pi, \operatorname{Im} z = 0 \} \end{aligned}$$

c) Sketch, in a single figure, the images under  $f$  of the following sets

$$\begin{aligned} & \{ z \in \mathbb{C} \mid \pi < \operatorname{Re} z \leq 2\pi, \operatorname{Im} z > 0 \} & \{ z \in \mathbb{C} \mid \pi < \operatorname{Re} z \leq 2\pi, \operatorname{Im} z < 0 \} \\ & \{ z \in \mathbb{C} \mid \operatorname{Re} z = \pi, \operatorname{Im} z > 0 \} & \{ z \in \mathbb{C} \mid \operatorname{Re} z = \pi, \operatorname{Im} z < 0 \} \\ & \{ z \in \mathbb{C} \mid \operatorname{Re} z = 2\pi, \operatorname{Im} z > 0 \} & \{ z \in \mathbb{C} \mid \operatorname{Re} z = 2\pi, \operatorname{Im} z < 0 \} \\ & \{ z \in \mathbb{C} \mid \pi < \operatorname{Re} z \leq 2\pi, \operatorname{Im} z = 0 \} \end{aligned}$$

d) Describe  $\Gamma$ , as viewed from the  $w$ -axis.

4) Let  $M$  and  $N$  be Riemann surfaces. Let  $f : M \rightarrow N$  be a nonconstant holomorphic map. Let  $P \in M$  and let  $\{P_{j,k}\}_{k \in \mathbb{N}}$ ,  $1 \leq j \leq n$ , be  $n$  sequences in  $M$  each obeying  $\lim_{k \rightarrow \infty} P_{j,k} = P$ . Furthermore, suppose that, for each  $k \in \mathbb{N}$ , the  $n$  points  $P_{1,k}, \dots, P_{n,k}$  are all distinct, but  $f(P_{1,k}) = f(P_{2,k}) = \dots = f(P_{n,k})$ . Prove that the branching number  $b_f(P)$  of  $f$  at  $P$  is at least  $n - 1$ .

5) Let  $a, b, c$  and  $d$  be four complex numbers with  $ad - bc \neq 0$ . If  $c \neq 0$ , define

$$g(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq -\frac{d}{c} \\ \infty & \text{if } z = -\frac{d}{c} \\ \frac{a}{c} & \text{if } z = \infty \end{cases}$$

and if  $c = 0$ , define

$$g(z) = \begin{cases} \frac{a}{d}z + \frac{b}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty \end{cases}$$

- a) Show that  $g(z)$  is a holomorphic map from  $\mathbb{C}_\infty$  to  $\mathbb{C}_\infty$ .
- b) Show that, if  $g$  is not the identity map, then  $g(z)$  has exactly two fixed points in  $\mathbb{C}_\infty$ , counting multiplicity.

6) Let  $D$  be a connected open set in  $\mathbb{C}_\infty$  and let  $G$  be a group of Möbius transformations, as in problem 5, such that

- 1)  $g(D) = D$  for all  $g \in G$
- 2) If  $g \in G$  and  $g$  is not the identity map, then  $g$  has no fixed point in  $D$ .
- 3) If  $K$  is a compact subset of  $D$  then  $\{g \in G \mid g(K) \cap K \neq \emptyset\}$  is finite.

- a) Prove that, for each  $z \in D$ , there is an open neighbourhood  $D_z$  of  $z$  in  $\mathbb{C}_\infty$  that is a subset of  $D$  and that obeys  $g(D_z) \cap D_z = \emptyset$  for all  $g \in G$  other than the identity map.
- b) Define, for  $z \in D$ ,  $[z] = \{g(z) \mid g \in G\}$ . Fix any  $z_1 \in D$ . Prove that the map  $z \mapsto [z]$  is 1-1 on  $D_{z_1}$ . Call this map  $\zeta_{z_1}^{-1}$ .
- c) Prove that  $\zeta_{z_1} \circ \zeta_{z_2}^{-1}$  is analytic on

$$\zeta_{z_2}(\zeta_{z_1}^{-1}(D_{z_1}) \cap \zeta_{z_2}^{-1}(D_{z_2})) = \{z \in D_{z_2} \mid \exists z' \in D_{z_1} \text{ with } [z] = [z']\}$$

Hint: If  $z \in D_{z_1}$  and  $z' \in D_{z_2}$  obey  $[z] = [z']$ , then there is a  $g \in G$  such that  $z' = g(z)$ .